A DUAL APPROACH TO THE IMPLEMENTATION OF A GENERAL EQUILIBRIUM MODEL

Hidetaka Kawano※

1. INTRODUCTION

It is well known that the primal and dual approaches are essentially looking at the same thing from different angles, and that a relationship which seems difficult to analyze from the primal approach often becomes very simple to analyze from the dual approach[1]. Samuelson's (1967) traditional primal approach suggests that equilibrium in the production sector of an economy is obtained as a solution to a usual maximization problem defined on the factor quantity space. Woodland (1977) equivalently formulated the same problem using a dual approach[2]. He formulated the equilibrium conditions for the production side of the economy in terms of cost functions rather than production functions on factor price space. Woodland's successful application of the duality principle to the production side of a general equilibrium model was the motivation in this paper to utilize the principle of duality in computing a general equilibrium model.

On the applied side of a general equilibrium model, the general equilibrium framework has been very intractable in dealing with multidimensional empirical issues. This is due to the lack of both efficiently operational algorithms and computational power. However, this kind of criticism no longer holds because of recent technical advancements in computer technology and subsequent refinements in operational algorithms. Scarf (1967 and 1973) in particular has made a significant contribution to the growth of applied general equilibrium modelling (also called computable general equilibrium modelling) as a useful framework for policy analysis, especially in the last 15 years[3].

In this paper, both duality between cost function and production function and duality

1) The principle of duality was first analyzed by Shepherd (1953) and later further analyzed by Uzawa (1964), Shepherd (1970), and Diewert (1971, 1974).

2) Woodland (1977) employed a dual approach in trade theory in order to focus on the analysis of commodity-factor price relationship such as the factor price equalization and the Stolper-Samuelson theorems.

3) For this contribution, Scarf was outstanding and he was awarded a distinguished fellowship of the American Economic Association in 1991.

4) Indirect utility function is derived, once the related expenditure function is estimated.

※Aomori Public College

between indirect utility function and (direct) utility function are utilized for computing general equilibrium solutions[4]. Specifically, a dual approach with an iterative output price revision rule for solving a general equilibrium model is discussed, and the strength of the approach is demonstrated with some general equilibrium applications[5]. Shoven and Whalley's (pp. 43-44, 1992) primal approach with a modified version of the iterative Kimbell-Harrison factor price revision rule (See Kawano, pp. 39-42, 2000) is in direct comparison to the dual approach with an iterative output price revision rule[6]. In Shoven and Whalley's primal approach, some form of the factor price revision rule must be used to solve a general equilibrium problem, since the entire model is reduced to the number of excess factor demand functions. However the factor price revision rule is not always convenient to implement, especially in the case of a small open economy with various tax structures. This is because an initial factor price must be given prior to the execution of the iterative factor price revisions. It is, therefore, impossible to assign the specific initial value for the relative world price of a commodity in the output market, although the assignment of the relative world price of a commodity is by construction required in the case of a small open economy model. While the case of a closed economy model can utilize either a primal approach or a dual approach, for a small open economy the dual approach is superior.

The main strengths of a dual approach in comparison with a primal approach are: (1) the derivations of the structural equations in the model are made simple by making use of Shephard's lemma and Roy's identity; (2) an output price revision rule, which is more convenient than a factor price revision rule for a small open economy model in particular, is easily implemented; and (3) the overall programming structure of the model is modified with relative ease. Because of these strengths, the dual approach has considerable merit in the implementation of large-scale empirical general equilibrium models.

In section 2, the general structure of the benchmark model is specified. In section 3, the dual approach to implementing the general equilibrium model is specified. In section 4, the existence, uniqueness and stability of the illustrative benchmark model are discussed. Some applications of the dual approach are discussed in section 5. The conclusion follows in section 6. All notations are defined as they appear for the first time in the text.

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5) All illustrative models are programmed in C-language and run by the Intel's 333MHz Pentium II Processor.

6) The original factor price revision rule is discussed by Kimbell and Harrison(1986). Shoven and Whalley (pp. 43-44, 1992) set out a computational solution procedure for a two-good-two factor general equilibrium model in a primal approach.

2. THE GENERAL STRUCTURE OF THE BENCHMARK MODEL

2.1. The main feature of the closed economy model

A simplified numerical general equilibrium model illustrates how the dual approach is implemented[7]. This simplified benchmark closed economy model has two final commodities (commodities 1 and 2) and two factors of production (capital K and labor L). The endowments of both factors are fixed in supply and both are owned by a representative consumer[8]. There are two producers: producers 1 and 2 producing the corresponding commodities 1 and 2. Both consumers and producers are assumed to act competitively in all commodity and factor markets. On the production side, production technology in each sector is represented by a constant-returns-to-scale, Cobb-Douglas production function. Each factor demand function is derived from a cost minimization problem subject to given technology and a given output level. On the demand side, the consumer's preference is represented by a Cobb-Douglas utility function. Each commodity demand function is derived from a utility maximization problem subject to the budget constraint faced by a consumer. There are two exogenous variables: the consumer endowments of labor and capital ($\bar{K}$and $\bar{L}$). There are seventeen endogenous variables for the required conditions for equilibrium, divided here into six main categories: (1) four prices (p_1, p_2 , w, r), (2) two commodities demanded (X_1 and X_2), (3) two commodities supplied (Q_1 and Q_2), (4) four factors demanded (K_1, K_2, L_1, L_2), (5) generated income (Y), and (6) four factors demanded per unit of output (k_1,k_2 , l_1, l_2). The closed economy model is summarized in Appendices C and D for the primal and dual approaches. The solution to the model characterized by the seventeen endogenous variables must satisfy the equilibrium conditions in the model: (1) excess demand conditions for all commodities and factors, and (2) zero-profit conditions in each industry. In addition, all excess demand functions for final commodities and factors are continuous, non-negative, homogenous of degree zero in their respective prices and must satisfy Walras' Law for theoretical consistency. Only relative prices are significant in affecting economic agents' decisions in general equilibrium models so that commodity price p_2 is chosen as numeraire. In programming the models, a dual approach is easier and sometimes more convenient than the primal approach.

2.2. The demand side of the model

There is only one representative consumer in the economy. The consumer maximizes his own utilities by solving the following constrained maximization :

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7) This approach will be considerably helpful in implementing many other large scale empirical models for various policy analyses.

8) The total gains for the economy is focused on in this model. Other important distributional issues are not considered here.

$$v(p_1, p_2, \bar{Y}) = \max_{[X_1,\, X_2 \geq 0]} U(X_1, X_2)$$
$$s.t. \sum_{i \in I} p_i X_i = \bar{Y} \tag{1}$$

where

$i \in I := \{1, 2\}$,

$\bar{L}$:= labor endowment for consumer ,

$\bar{K}$:= capital endowments for consumer ,

w := wage rate,

r := rental rate,

p_i := commodity price $i \in I$,

$\bar{Y}$:= given level of income for consumer,

Remark: If the values of w and r change, then the income of consumer is expressed as $Y = wL + rK$.

X_i := ith commodity demanded $i \in I := \{1, 2\}$,

$U(.)$:= a well behaved standard neoclassical utility function,

$v(.)$:= the corresponding indirect utility function to the direct utility function $U(.)$.

The commodities demanded, X_1 and X_2, are the solutions to the maximization problem (1). The demand function X_i is expressed as:

$$X_{i \in I} = X_{i \in I}(p_1, p_2, \bar{Y}) \tag{2}$$

First substitute the derived demand functions $X_{i \in I}$ into the objective function in problem (1) and eliminate the constants to derive the indirect utility function. In the case of Cobb-Douglas direct utility function $U(.)$:

$$U(X_1, X_2) = X_1^{a_1} X_2^{a_2}, \qquad \because a_1 + a_2 = 1, \tag{3}$$

where

a_i := a parameter.

The corresponding indirect utility function $v(.)$ is:

$$v(p_1, p_2, Y) = \ln Y - a_1 \ln p_1 - a_2 \ln p_2, \qquad \because a_1 + a_2 = 1. \tag{4}$$

The Marshallian demand functions derived by Roy's identity are:

$$X_i = \frac{-\dfrac{\partial v(p_1, p_2, Y)}{\partial p_i}}{\dfrac{\partial v(p_1, p_2, Y)}{\partial Y}} = \frac{a_i Y}{p_i}, \qquad \forall\, i = 1, 2. \tag{5}$$

Another efficient way to derive the indirect utility function is to make use of the Cobb-Douglas type of expenditure function. In the principle of duality, all three functions: the direct utility function, the indirect utility function, and the expenditure function, contain essentially the same information about consumer behavior.

2.3. The production side of the model

There are two industries ($i \in I := \{1, 2\}$). Aggregate (constant returns to scale) industry production functions for both industries are assumed. An aggregate producer in each perfectly competitive industry maximizes his profit by solving the following constrained cost-minimization problem, subject to both given technology and given output level $\bar{Q}_i$ which is iteratively revised to reach the competitive general equilibrium :

$$\begin{aligned} C_i(r, w, \bar{Q}) &= \min_{[L_i,\, K_i \geq 0]} \; C_i(L_i, K_i) = w L_i + r K_i \\ s.t. \quad & Q_i(L_i, K_i) = \bar{Q}_i, \qquad \forall\, i \in I. \end{aligned} \tag{6}$$

where

$C_i\,(.) :=$ (direct) cost function,

$L_i :=$ labor demand,

$K_i :=$ capital demand,

$\bar{Q}_i :=$ given output level,

$Q_i\,(.) :=$ a well behaved neoclassical (strictly quasiconcave) production function for the ith aggregate industry production function $i \in I$.

The derived factor (labor and capital) demands for given output level $\bar{Q}_i$ are :

$$L_i = L_i(r, w, \bar{Q}_i), \qquad K_i = K_i(r, w, \bar{Q}_i). \tag{7}$$

The corresponding cost function to the minimization problem (6) is derived by substituting the derived factor demand functions in equation (7) into the objective function. In the case of Cobb-Douglas production function:

$$Q_i = K_i^{\alpha_i} L_i^{1-\alpha_i}, \qquad \forall\, i = 1, 2. \tag{8}$$

where

α_i:= a factor share parameter.

The corresponding cost function is:

$$C_i(r, w, \bar{Q}) = A_i r^{\alpha_i} w^{1-\alpha_i} \bar{Q}, \quad \because A_i = \alpha_i^{-\alpha_i}(1-\alpha_i)^{\alpha_i - 1} \quad \forall\, i = 1, 2. \tag{9}$$

The unit cost function, which is positive, linearly homogeneous, and a concave function of factor prices, is:

$$c_i(r, w) = A_i r^{\alpha_i} w^{1-\alpha_i}, \quad \because A_i = \alpha_i^{-\alpha_i}(1-\alpha_i)^{\alpha_i - 1} \quad \forall\, i = 1, 2. \tag{10}$$

The unit factor demand functions k_i and l_i derived by Shephard's lemma are:

$$k_i(r, w) = \frac{\partial c_i(r, w)}{\partial r} = A_i \alpha_i \left(\frac{w}{r}\right)^{1-\alpha_i}, \qquad \forall\, i = 1, 2. \tag{11}$$

and

$$l_i(r, w) = \frac{\partial c_i(r, w)}{\partial w} = A_i (1-\alpha_i) \left(\frac{w}{r}\right)^{-\alpha_i}, \qquad \forall\, i = 1, 2. \tag{12}$$

The closed economy benchmark model structures for the primal and dual approaches are summarized in Appendices C and D.

2.4. Excess demand conditions

Excess demand conditions for both commodities and factors are :

$$\begin{aligned} &X_i(p_1, p_2, Y) - Q_i \le 0 \qquad \forall\, i \in I. \\ &\sum_{i \in I} L_i(r, w, Q_i) - \bar{L} \le 0. \\ &\sum_{i \in I} K_i(r, w, Q_i) - \bar{K} \le 0. \end{aligned} \tag{13}$$

In the dual approach, only commodity price p_1 is iteratively revised, given commodity price p_2 as a numeraire ($p_2 = 1$). Therefore, the excess demand conditions stated above are simply rewritten as:

$$X_i(p_1) - Q_i(p_1) \leq 0 \quad \forall\, i \in I.$$

$$\sum_{i \in I} L_i(p_1) - \bar{L} \leq 0. \tag{14}$$

$$\sum_{i \in I} K_i(p_1) - \bar{K} \leq 0.$$

2.5. Zero-profit conditions

If the output of industry i is positive, the price of output i is equal to the long run average costs (zero-profit in the long run) under perfect competition.

$$p_i = \frac{C_i\big(L_i(r, w, Q_i), K_i(r, w, Q_i)\big)}{Q_i}$$

$$p_i = \frac{w L_i(r, w, Q_i) + r K_i(r, w, Q_i)}{Q_i} \tag{15}$$

$$p_i = w\, l_i(r, w) + r K_i(r, w).$$

Equivalently, the average-cost (zero profit) conditions are expressed by the unit cost function as:

$$p_i = c_i(r, w)$$

$$= A_i r^{\alpha_i} w^{1-\alpha_i}, \quad \because A_i = \alpha_i^{-\alpha_i}(1-\alpha_i)^{\alpha_i - 1} \quad \forall\, i = 1, 2. \tag{16}$$

2.6. Walras' law

Finally, any set of prices for a general equilibrium model must satisfy Walras' law. In other words, every iterative revision of commodity and factor prices must satisfy the following equation for theoretical consistency. In the dual approach, only commodity price p_1 is iteratively revised, given commodity price p_2 as a numeraire (p_1=1). In addition, factor prices w and r are adjusted to clear their respective factor markets: both labor and capital markets in every iteration that commodity price p_1 is revised so that Walras' law is simply stated as:

$$p_1\left[X_1(p_1) - Q_1(p_1)\right] + \left[X_2(p_1) - Q_2(p_1)\right] = 0, \quad \because p_2 = 1. \tag{17}$$

2.7. *Specification of parameters and exogenous variables*

In this simple closed economy model, there are four parameter values to be specified: (1) two production function parameters for two commodities supplied by two industries, and (2) two utility function parameters for two commodities demanded. There are two exogenous variables: the endowment of labor and capital for the consumer. All specified parameter and exogenous values are summarized in table 1.

TABLE 1

Specification of parameters and exogenous variables for a simple general equilibrium model

Production Parameters	Demand Parameters	Factor Endowments
$\alpha_1 = 0.1$	$a_1 = 0.5$	$\bar{K} = 0.8$
$\alpha_2 = 0.9$	$a_2 = 0.5$	$\bar{L} = 2.0$

3. THE DUAL APPROACH TO IMPLEMENTING THE GENERAL EQUILIBRIUM MODEL

3.1. *A closed economy model*

The dual approach with the iterative output price revision rule is presented in this section. Shoven and Whalley's (pp. 43-44, 1992) primal approach with the iterative modified Kimbell-Harrison factor price revision rule discussed in Kawano (2000) can also be applied to solve the same general equilibrium problem[9]. Shoven and Whalley (pp. 43-44, 1992) set out a simple computational solution procedure for a two-good-two-factor general equilibrium model in the traditional primal approach[10]. In their primal approach, some

9) The original Kimbell-Harrison type of factor price revision rule (1986) is simplified in Kawano (2000) by assuming that the weighted average of the elasticities of substitution proposed by Kimbell and Harrison (1986) is equal to unity. As noted by Kimbell and Harrison (1986), this factor price revision rule is a simple form of the Walrasian tâtonnement process that raises the price of a factor in excess demand and lowers the price of a factor in excess supply. Samuelson(1947) formulated the simultaneous Walrasian tâtonnement process as a set of differential equations, describing the price changes of each commodity in proportion to its excess demand at any time (Arrowand Hurwicz, 1958).

10) The procedure is to reduce the dimensionality of solution space to the number of factors of production. In other words, the equilibria for this two-good-two-factor model are characterized by two excess factor demand functions for both capital and labor. Either wage rate or rental rate can be considered as a numeraire, since relative prices are important in a general equilibrium setting. Due to Walras' law, the entire general equilibrium system is collapsed to a single equation to solve for the optimal rental rate r^*. To find a root r^* of the single equation, many varieties of algorithms can also be applied to this solution procedure. In this approach, any good choice of algorithm to locate roots of equations, such as Bisection method, many Newton and Secant method varieties, etc. (Tanaka and Kawano, 1996), can be applied to find any real number root r^* for which $f(r^*) = 0$.

form of the factor price revision rule must be used to solve the general equilibrium problem. However, Shoven and Whalley's primal approach with the factor price revision rule is not always convenient to implement, especially for a small open economy model with various tax structures, since the entire model is reduced to the number of excess factor demand functions. However the dual approach with the output price revision rule can be easily implemented and modified. For comparison the closed economy model structures used for both primal and dual approaches are summarized in Appendices C and D. The small open economy model structures used for both approaches are summarized in Appendices E and F. For purposes here, first, the solution procedure of the dual approach with the output price revision rule for a closed economy model is specified. Then, the case of a small open economy model follows in subsection 3.2.

Step 1: Choose commodity price p_2 as a numeraire (p_2=1) and assign the arbitrarily chosen strictly positive initial value to commodity price p_1=0.152 ($\because p_1>0$).

Step 2: Express factor prices: wage rate w and rental rate r ($\because w, r>0$) as a function of commodity prices p_1 and p_2.

(2-1) Derive the unit cost functions.

$$c_i(r, w) = A_i r^{\alpha_i} w^{1-\alpha_i}, \quad \because A_i = \alpha_i^{-\alpha_i}(1-\alpha_i)^{\alpha_i - 1} \quad \forall\, i = 1, 2 \tag{18}$$

(2-2) Rewrite the average-cost pricing (zero profit) conditions:

$$p_i = c_i(r, w), \qquad \forall\, i = 1, 2 \tag{19}$$

as the following two functions:

$$p_1 = A_1 r^{\alpha_1} w^{1-\alpha_1}, \quad \because A_1 = \alpha_1^{-\alpha_1}(1-\alpha_1)^{\alpha_1 - 1} \tag{20}$$

and

$$p_2 = A_2 r^{\alpha_2} w^{1-\alpha_2}, \quad \because A_2 = \alpha_2^{-\alpha_2}(1-\alpha_2)^{\alpha_2 - 1} \tag{21}$$

(2-3) Solve the above two average-cost pricing equations (20) and (21) for both wage rate w and rental rate r. The wage rate w is expressed as a function of commodity prices p_1 and p_2.

$$w(p_1, p_2) = \left((p_1 A_1^{-1})^{\alpha_2/\alpha_1} p_2^{-1} A_2 \right)^{\frac{\alpha_1}{\alpha_1(\alpha_2 - 1) - (\alpha_1 - 1)\alpha_2}} \tag{22}$$

The rental rate r is also expressed as a function of commodity prices p_1 and p_2.

$$r(p_1, p_2) = \left(p_1 A_1^{-1} w(p_1, p_2)^{\alpha_1 - 1}\right)^{\frac{1}{\alpha_1}} \tag{23}$$

Step 3: Compute the income using the endowments of fixed labor and capital, $\bar{L}$ and $\bar{K}$.

$$Y = r\bar{K} + w\bar{L} \tag{24}$$

Step 4: Derive individual commodity demands X_1 and X_2 through the Roy's identity given the indirect utility function (the indirect objective function of the utility maximization problem), then compute the values of the commodity demand functions since commodity prices p_1, and p_2 are given in step 1, and income Y in step 3.

$$X_i = \frac{-\dfrac{\partial v(p_1, p_2, Y)}{\partial p_i}}{\dfrac{\partial v(p_1, p_2, Y)}{\partial Y}} = \frac{a_i Y}{p_i}, \quad \forall\, i = 1, 2. \tag{25}$$

Step 5: Determine factor demands per unit of output i through the Shephard's lemma, given a unit cost function (the indirect objective function of cost minimization problem) in step (2-1).

(5-1) Labor demand function per unit of output i is:

$$l_i = \frac{\partial c_i(r, w)}{\partial w} = A_i(1 - \alpha_i)\left(\frac{w}{r}\right)^{-\alpha_i}, \quad \forall\, i = 1, 2. \tag{26}$$

(5-2) Capital demand function per unit of output i is:

$$k_i = \frac{\partial c_i(r, w)}{\partial r} = A_i \alpha_i \left(\frac{w}{r}\right)^{1 - \alpha_i}, \quad \forall\, i = 1, 2. \tag{27}$$

Step 6: Derive commodity output Q_i[11].

(6-1) Capital demand for output Q_i is:

11) Commodity output Q_i should not be equated with commodity demand X_i in the process of revising commodity price p_1 and Q_i are X_i are equated only when the entire economic system is in equilibrium.

$$K_i = k_i Q_i, \qquad \forall\, i = 1, 2. \tag{28}$$

(6-2) Labor demand L_i for output Q_i is:

$$L_i = l_i Q_i, \qquad \forall\, i = 1, 2. \tag{29}$$

(6-3) Capital market clearing condition is:

$$K_1 + K_2 = \bar{K} \tag{30}$$

(6-4) Labor market clearing condition is:

$$L_1 + L_2 = \bar{L} \tag{31}$$

(6-5) By substituting equation (28) into (30) and equation (29) into (31), derive the following set of linear equations in a matrix form as:

$$\begin{bmatrix} l_1 & l_2 \\ k_1 & k_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \bar{L} \\ \bar{K} \end{bmatrix} \tag{32}$$

(6-6) Solve the above system of linear equations for commodity outputs Q_1 and Q_2 by Cramer's rule[12].

Output Q_1 is:

$$Q_1 = \frac{1}{H} \begin{vmatrix} \bar{L} & l_2 \\ \bar{K} & k_2 \end{vmatrix}, \qquad \because H = \begin{vmatrix} l_1 & l_2 \\ k_1 & k_2 \end{vmatrix} \tag{33}$$

Output Q_2 is:

12) For the large system of linear equations used in many extended models, a computational method, such as the relatively efficient LU-decomposition routine, is preferable to compute the solutions.

$$Q_2 = \frac{1}{H}\begin{vmatrix} l_1 & \bar{L} \\ k_1 & \bar{K} \end{vmatrix}, \quad \because H = \begin{vmatrix} l_1 & l_2 \\ k_1 & k_2 \end{vmatrix} \tag{34}$$

Step 7: Set up the crucial procedure for revising commodity price p_1 in iterations from the given initial value in step 1. Commodity price p_2 is treated as a numeraire (p_2=1). The converged value p_1 is achieved through a finite number of iterations $p_n=p_{n+1}=p$. The formulation for revising a commodity output price is specified as:

$$p_{1,n+1} = p_{1,n}\left(\frac{X_{1,n}}{Q_{1,n}}\right), \tag{35}$$

where

$p_{1,n}$:= finite n-th iterated commodity price 1,

$p_{1,n+1}$:= finite n-1 iterated commodity price 1,

$X_{1,n}$:= finite n-th iterated quantity demanded for commodity 1,

$Q_{1,n}$:= finite n-th iterated quantity supplied for commodity 1.

This is a very simple form of the Walrasian tâtonnement process, which raises the price of a commodity in excess demand and lowers the price of a commodity in excess supply. This formulation is numerically much easier to solve than Samuelson's specification of the tâtonnement process in the form of a set of differential equations (Samuelson, 1947). The equilibrium commodity price p_1 is achieved when a commodity 1's demand and supply are equated at a specified level of tolerance.

Step 8: Specify the level of tolerance for the relative error form of the capital market clearing condition to be satisfied. The relative error is desired for the accuracy of an approximation (Higham, p.5, 1996).

The relative error form of the condition is specified as:

$$\left|\frac{X_1 - Q_1}{X_1}\right| > \mathit{Tolerance} = 1.0e\text{-}15\ . \tag{36}$$

Step 9: Repeat the entire process from steps 1 to 8 and keep revising the finite n-th iterated commodity price $p_{1,n}$ until its convergence occurs with the above condition in step 8 satisfied. In this benchmark model, only the 24th iterative commodity 1's price revisions were executed by the Intel's 333MHz Pentium II processor. The computational time could not be measured with the precision of 1/1000 seconds (see lines 33-34 in Appendix G). The computational results are shown in Figure A-1 in Appendix A and in lines 1-34 in Appendix G.

3.2. A small open economy model

For the small open economy model, given commodity price as a numeraire (p_1=1), the arbitrarily chosen strictly positive initial value to a commodity price p_1 ($\because p_1>0$) does not have to be revised. The initial value is chosen as the world (relative) price that a small open country is facing in the world market. Therefore, the entire solution procedure is much simpler than the case of the closed economy since the initially chosen commodity price p_1 does not require the iterative revision. Another difference between the small open economy model and the closed economy model is the inclusion of import and export equations in market clearing conditions in equations 6-7 in Appendix F. It is noteworthy that the trade balance condition has been already implied by both producer and consumer optimization conditions. In other words, the trade balance condition is redundant in programming the structure of a small open economy. The solution procedure for a small open economy model is:

Step 1: Choose commodity price p_2 as a numeraire (p_2=1) and assign the arbitrarily chosen strictly positive initial value to a commodity price p_1=0.152 ($\because p_1>0$), given the world market price that this small open economy is facing.

Step 2: Follow exactly the same procedure in steps 2-6 as in the case of the closed economy. Once the execution of the program reaches step 6, the program ends, since the commodity price (the world price) p_1 is given at step 1 and is not to be revised. The computational results are shown in Figure A-2 in Appendix A and in lines 1-44 in Appendix H.

4. EXISTENCE, UNIQUENESS AND STABILITY

For theoretical consistency and reliable simulation results, it was crucial to investigate the existence, uniqueness and the stability of the equilibrium in the price adjustment process[13]. Instead of a formal axiomatic analyses, a computational approach to this issue

13) In formal axiomatic analysis, Arrow, Block, and Hurwicz (1959) and Arrow and Hahn (1971) obtained a major analytical result on the stability of a general equilibrium model.

can be employed. Although a numerical computational approach is not a rigorous proof, it can be considered a close approximation to a formal axiomatic analysis. For the ease of exposition, I used the benchmark closed economy model to investigate the existence, the uniqueness and the stability of the equilibrium in the price adjustment process.

First, define the following function $F(p_1)$ as a price adjustment rule:

$$F(p_1) \equiv p_1 \left(\frac{X_1(p_1)}{Q_1(p_1)} \right), \tag{37}$$

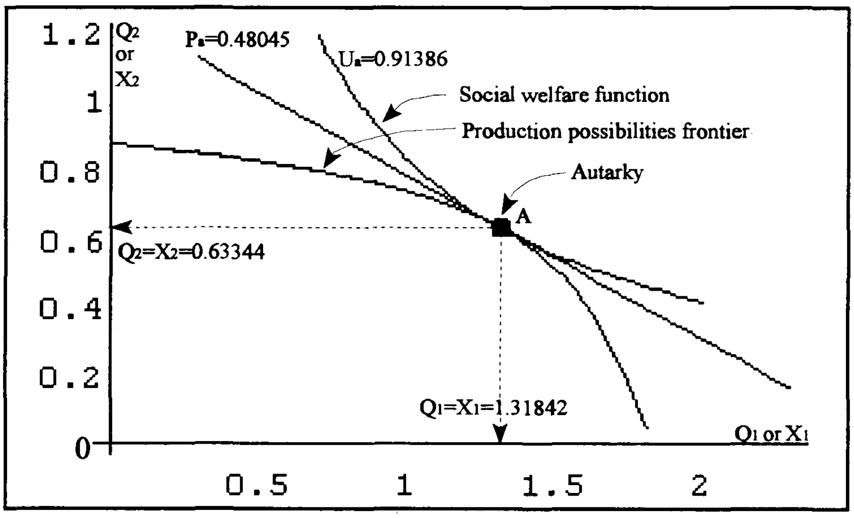
The right-hand side of the function is formally defined as a commodity price revising rule for price p_1. This price adjustment rule is also a very simple form of the Walrasian tâonnement process as noted in section 3. The price space is now confined to the unit simplex for convenience without loss of generality, because only the relative prices matter in a general equilibrium analysis. Since the price space is defined as:

$$P = \left\{ p \mid p \in R^2_{++}, \sum_{p_i=1}^{2} p_i = 1 \right\}, \tag{38}$$

the Brouwer fixed-point theorem is stated as: *Let F(.) be a continuous function, F:P→P. Then there is some* $p^* \in p$ *so that* $F(p^*)=p^*$. The computed equilibrium price vector is the only price vector $(p_1, p_2) = (0.324529600505579, 0.675470399494421)$ that satisfy the equation:$F(p^*)=p^*$. The price vector also satisfies the imposed unit simplex condition: $p_1+p_2=1$. The relative price $p_a=p_1/p_2=$ 0.480449773592573 is also computed and shown in Figure 1[14]. The powerful Brouwer fix-point theorem assures the existence of the general equilibrium price vector.

14) The production possibilities frontier (the locus of efficient outputs for the economy) in Figure 1 is iteratively computed. This computation assumes the relative price of commodity 1 is sequentially changed. This in done since the function for the frontier is not given numerically in a closed form (it is generally shown to be a strictly convex function of commodity outputs 1 and 2).

Figure 1.



For the closed economy's case presented in Appendix A, commodity price p_2 is a numeraire. The same unique relative price $p_a = p_1/p_2 =$ 0.480449773592573 is also computed as above, in that the price space is confined on the unit simplex. The convergence occurs at the specified level of tolerance, regardless of the initial values assigned. Thus, both computational rule and the unique equilibrium could be claimed to be globally stable.

5. SOME APPLICATIONS OF THE DUAL APPROACH

The main strengths of a dual approach in comparison with a primal approach are: (1) the derivations of the structural equations in the model are made simple by making use of both Shephard's lemma and Roy's identity; (2) an output price revision rule, which is more convenient than a factor price revision rule particularly for a small open economy model, is easily implemented; and (3) the overall programming structure of the model is modified with relative ease. Because of these strengths, in this section, I describe some of the applications of the dual approach with the iterative output price revision rule to some general equilibrium models: 1) the benchmark closed economy, 2) a small open economy, 3) a small open economy with production tax, and 4) a small open economy with consumption tax. These four models are successfully executed in using the dual approach. This approach is of considerable merit in implementing large-scale empirical general equilibrium models. Of theoretical importance, the existence of a competitive equilibrium with taxes is essential to reliable simulation results, as proven by Shoven and Whalley (1973). Earlier proof of the existence of competitive equilibria in international trade models was obtained by Sonthimer (1971) and Shoven (1974). Without such a sound theoretical framework, any numerical simulation of general equilibria may not be persuasive and cannot be seriously considered in terms of implementing policy.

First, the benchmark closed economy is simulated. The model is a simple illustration of combining the production and demand sides of the economy to arrive at a general equilibrium. This economy is self-sufficient and is isolated from the rest of the world. There is no international trade. The model is illustrated in Figure A-1, Appendix A. On the demand side of the model, the representative consumer's marginal rate of substitution (MRS) is equal to the relative price ratio p_a. The consumer consumes optimally at point A. On the production side, the marginal rate of transformation (MRT) is equal to the relative price ratio p_a. Producers produce optimally at point A. The economic outcome is efficient since the highest utility U_a is attained on the production possibilities frontier at point A. The numerical results are provided in Appendix G.

Second, the case of a small open economy is simulated. The economy can engage in international trade with the rest of the world at a given relative world price p_W. The trading economy is no longer constrained to consume only what it can produce. This point is illustrated in Figure A-2, Appendix A. The trade balance condition implies that the value of her imports should be equal to the value of exports, which is shown in lines 38-39, Appendix H. The welfare of the open economy is improved by trading with others, because a higher utility is attained at consumption point A, separate from production point Q. As a result of trading with the rest of the world, more consumption bundles are available, also more specialization takes place towards production with the usual Ricardo's comparative advantage gain from trade, than in a closed economy.

Third, a small open economy with a production tax on commodity 2 is simulated. This example is taken from Markusen et al. (1995). For simplicity, the given world price faced by the small open country happens to be equal to the country's autarky price at point A. The model is illustrated in Figure B-1 in Appendix B. In this case, the relative producer price deviates from the relative world price. The consumer price is identical to the world price. The production tax discourages production of commodity 2 and encourages production of commodity 1. The national budget constraint is reduced, and as a result consumption of both commodities are reduced, compared with the situation before the production tax is imposed. The general equilibrium effect of production tax policy is national welfare reduction. This is shown in Figure B-1, Appendix B, and also in lines 45-46, Appendix I.

Fourth, a small open economy with a consumption tax on commodity 2 is simulated. This example is also taken from Markusen et al. (1995). The consumption tax does not affect production at point A. The tax levied on commodity 2 discourages a consumer from purchasing commodity 2, because of its increased price. On the other hand, the same tax encourages a consumer to purchase the non-taxed commodity 1. Welfare is reduced because the consumer's choice is distorted by the consumption tax. This is illustrated in Figure B-2, Appendix B. All computations results are in Appendix J.

6. CONCLUSION

The primal and dual approaches are essentially looking at the same thing from different angles. In this paper, duality between cost function and production function and duality between indirect utility function and (direct) utility function are utilized for computing general equilibrium solutions. Specifically, a dual approach with an iterative output price revision rule for solving a general equilibrium model is discussed, and the strength of the approach is demonstrated using general equilibrium applications. While a closed economy model can utilize either a primal approach with a factor price revision rule or a dual approach with an output price revision rule, for a small open economy, the dual approach is superior. The main strengths of a dual approach in comparison with a primal approach are: (1) the derivations of the structural equations in the model are made simple by making use of Shephard's lemma and Roy's identity; (2) an output price revision rule, which is more convenient than a factor price revision rule for a small open economy model in particular, is easily implemented; and (3) the overall programming structure of the model is modified with relative ease. Because of these strengths, the dual approach is highly useful in implementing large-scale empirical general equilibrium models.

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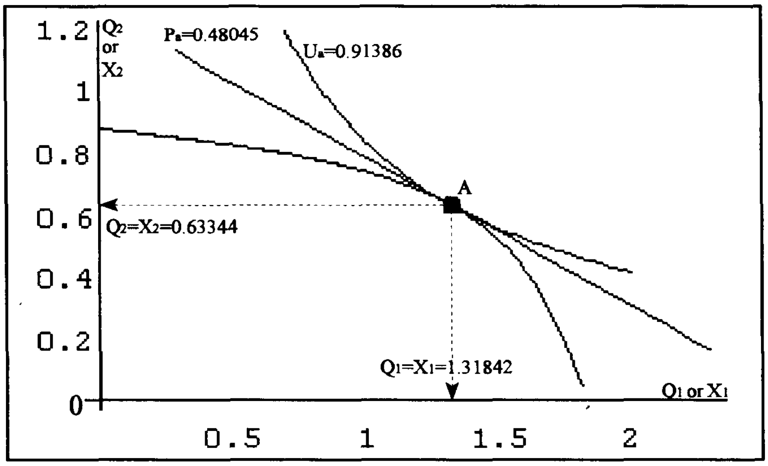
Appendix

APPENDIX A

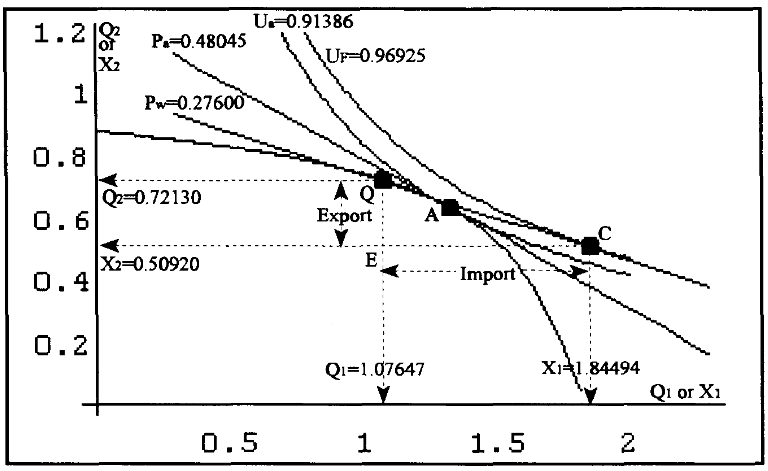
1) A closed economy

Figure A-1



2) A small open economy

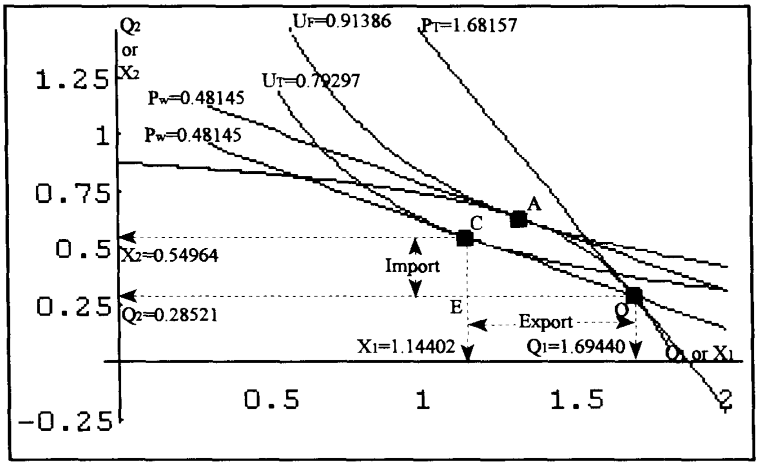
Figure A-2



APPENDIX B

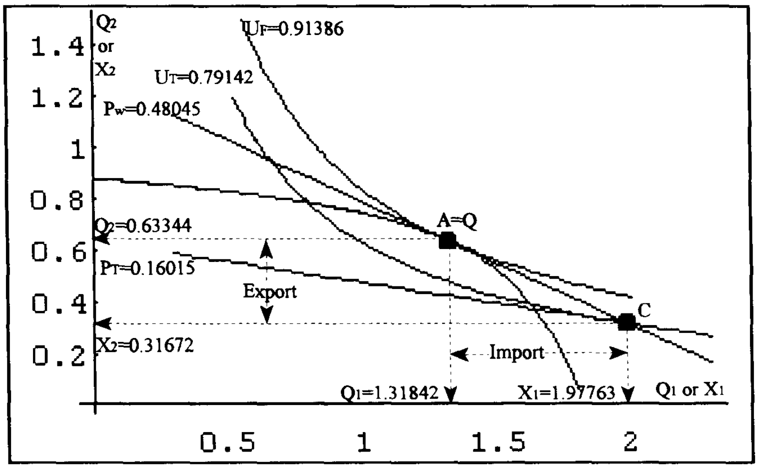
1) A small open economy with production tax on Q_2

Figure B-1



2) A small open economy with consumption tax on Q_2

Figure B-2



APPENDIX C

A primal approach for a closed economy

COMMODITY MARKETS

Utility function:

$$U(X_1, X_2) = X_1^{a_1} X_2^{a_2}, \qquad \because a_1 + a_2 = 1.$$

Production function:

$$Q_i = K_i^{\alpha_i} L_i^{1-\alpha_i}, \qquad \forall\, i = 1, 2.$$

Consumer's income:

$$Y = r\bar{K} + w\bar{L} \tag{1}$$

Demand:

$$X_i = \frac{a_i Y}{p_i}, \qquad \forall\, i = 1, 2. \tag{2)-(3}$$

Average-cost pricing (zero profit) conditions:

$$p_i = rk_i + wl_i, \qquad \forall\, i = 1, 2. \tag{4)-(5}$$

Market clearing:

$$Q_i = X_i, \qquad \forall\, i = 1, 2. \tag{6)-(7}$$

FACTOR MARKETS

Unit factor demand:

$$k_i = \left(\frac{\alpha_i}{1-\alpha_i}\right)^{1-\alpha_i} \left(\frac{w}{r}\right)^{1-\alpha_i}, \qquad \forall\, i = 1, 2. \tag{8)-(9}$$

$$l_i = \left(\frac{\alpha_i}{1-\alpha_i}\right)^{-\alpha_i} \left(\frac{w}{r}\right)^{-\alpha_i}, \qquad \forall\, i = 1, 2. \tag{10)-(11}$$

$$K_i = k_i Q_i, \qquad \forall\, i = 1, 2. \tag{12)-(13}$$

$$L_i = l_i Q_i, \qquad \forall\, i = 1, 2. \tag{14)-(15}$$

Market Clearing:

$$K_1 + K_2 = \bar{K} \tag{16}$$

$$L_1 + L_2 = \bar{L} \tag{17}$$

VARIABLES IN THE MODEL

The 17 endogenous variables:

$X_1, X_2, Q_1, Q_2, K_1, K_2, L_1, L_2,$
$k_1, k_2, l_1, l_2, P_1, P_2, w, r, Y.$

The 2 exogenous variables:

$\bar{K}, \bar{L}.$

The 4 parameters

$\alpha_1, \alpha_2, a_1, a_2.$

APPENDIX D

A dual approach for a closed economy

DOMESTIC GOODS MARKETS

Indirect utility function:

$$v(p_1, p_2, Y) = \ln Y - a_1 \ln p_1 - a_2 \ln p_2$$

Unit cost function:

$$c_i(r, w) = A_i r^{\alpha_i} w^{1-\alpha_i}, \quad \because A_i = \alpha_i^{-\alpha_i}(1-\alpha_i)^{\alpha_i - 1} \quad \forall\, i = 1, 2.$$

Consumer's income:

$$Y = r\bar{K} + w\bar{L} \qquad (1)$$

Marshallian demand functions:

By Roy's identity,

$$X_i = \frac{-\dfrac{\partial v(p_1, p_2, Y)}{\partial p_i}}{\dfrac{\partial v(p_1, p_2, Y)}{\partial Y}} = \frac{a_i Y}{p_i}, \quad \forall\, i = 1, 2. \qquad (2)\text{-}(3)$$

Average-cost pricing (zero profit) conditions:

$$p_i = c_i(r, w), \quad \forall\, i = 1, 2. \qquad (4)\text{-}(5)$$

Market clearing conditions:

$$Q_i = X_i, \quad \forall\, i = 1, 2. \qquad (6)\text{-}(7)$$

DOMESTIC FACTOR MARKETS

Unit factor demand functions:

By Shephard's lemma,

$$k_i = \frac{\partial c_i(r, w)}{\partial r} = A_i \alpha_i \left(\frac{w}{r}\right)^{1-\alpha_i}, \quad \forall\, i = 1, 2. \qquad (8)\text{-}(9)$$

By Shephard's lemma,

$$l_i = \frac{\partial c_i(r, w)}{\partial w} = A_i (1-\alpha_i) \left(\frac{w}{r}\right)^{-\alpha_i}, \quad \forall\, i = 1, 2. \qquad (10)\text{-}(11)$$

$$K_i = k_i Q_i, \quad \forall\, i = 1, 2. \qquad (12)\text{-}(13)$$

$$L_i = l_i Q_i, \quad \forall\, i = 1, 2. \qquad (14)\text{-}(15)$$

Factor market clearing conditions:

$$K_1 + K_2 = \bar{K} \qquad (16)$$

$$L_1 + L_2 = \bar{L} \qquad (17)$$

VARIABLES IN THE MODEL

The 17 endogenous variables:

$X_1, X_2, Q_1, Q_2, K_1, K_2, L_1, L_2,$
$k_1, k_2, l_1, l_2, P_1, P_2, w, r, Y.$

The 2 exogenous variables:

$\bar{K}, \bar{L}.$

The 4 parameters

$\alpha_1, \alpha_2, a_1, a_2.$

APPENDIX E

A primal approach for a small open economy

COMMODITY MARKETS

Utility function:

$$U(X_1, X_2) = X_1^{a_1} X_2^{a_2}, \qquad \because a_1 + a_2 = 1.$$

Production function:

$$Q_i = K_i^{\alpha_i} L_i^{1-\alpha_i}, \qquad \forall\, i = 1, 2.$$

Consumer's income:

$$Y = r\bar{K} + w\bar{L} \qquad (1)$$

Demand:

$$X_i = \frac{a_i Y}{p_i}, \qquad \forall\, i = 1, 2. \qquad (2)\text{-}(3)$$

Average-cost pricing (zero profit) conditions:

$$p_i = rk_i + wl_p \qquad \forall\, i = 1, 2. \qquad (4)\text{-}(5)$$

Market clearing:

$$X_1 = Q_1 - E \qquad (6)$$
$$X_2 = Q_2 + M \qquad (7)$$

FACTOR MARKETS

Unit factor demand functions:

$$k_i = \left(\frac{\alpha_i}{1-\alpha_i}\right)^{1-\alpha_i} \left(\frac{w}{r}\right)^{1-\alpha_i}, \qquad \forall\, i = 1, 2. \qquad (8)\text{-}(9)$$

$$l_i = \left(\frac{\alpha_i}{1-\alpha_i}\right)^{-\alpha_i} \left(\frac{w}{r}\right)^{-\alpha_i}, \qquad \forall\, i = 1, 2. \qquad (10)\text{-}(11)$$

$$K_i = k_i Q_i, \qquad \forall\, i = 1, 2. \qquad (12)\text{-}(13)$$
$$L_i = l_i Q_i, \qquad \forall\, i = 1, 2. \qquad (14)\text{-}(15)$$

Market Clearing:

$$K_1 + K_2 = \bar{K} \qquad (16)$$
$$L_1 + L_2 = \bar{L} \qquad (17)$$

FOREIGN SECTOR

Price equations:

$$P_1 = \bar{P}_{W1} \qquad (18)$$
$$P_2 = \bar{P}_{W2} \qquad (19)$$

(Implied trade balance condition):

$$\bar{P}_{W1}\, E - \bar{P}_{W2}\, M = 0$$

VARIABLES IN THE MODEL

The 19 endogenous variables:

$X_1, X_2, Q_1, Q_2, K_1, K_2, L_1, L_2,$
$k_1, k_2, l_1, l_2, P_1, P_2, w, r, Y, E, M.$

The 4 exogenous variables:

$\bar{K}, \bar{L}, \bar{P}_{W1}, \bar{P}_{W2}.$

The 4 parameters $\alpha_1, \alpha_2, a_1, a_2.$

APPENDIX F

A dual approach for a small open economy

DOMESTIC GOODS MARKETS

Indirect utility function:

$$v(p_1, p_2, Y) = \ln Y - a_1 \ln p_1 - a_2 \ln p_2$$

Unit cost function:

$$c_i(r, w) = A_i r^{\alpha_i} w^{1-\alpha_i}, \quad \because A_i = \alpha_i^{-\alpha_i}(1-\alpha_i)^{\alpha_i - 1} \quad \forall\, i = 1, 2.$$

Consumer's income:

$$Y = r\bar{K} + w\bar{L} \qquad (1)$$

Marshallian demand functions:

By Roy's identity,

$$X_i = \frac{-\dfrac{\partial v(p_1, p_2, Y)}{\partial p_i}}{\dfrac{\partial v(p_1, p_2, Y)}{\partial Y}} = \frac{a_i Y}{p_i}, \quad \forall\, i = 1, 2. \qquad (2)\text{-}(3)$$

Average-cost pricing (zero profit) conditions:

$$p_i = c_i(r, w), \quad \forall\, i = 1, 2. \qquad (4)\text{-}(5)$$

Market clearing conditions:

$$X_1 = Q_1 - E \qquad (6)$$
$$X_2 = Q_2 + M \qquad (7)$$

DOMESTIC FACTOR MARKETS

Unit factor demand functions:

By Shephard's lemma,

$$k_i = \frac{\partial c_i(r, w)}{\partial r} = A_i \alpha_i \left(\frac{w}{r}\right)^{1-\alpha_i}, \quad \forall\, i = 1, 2. \qquad (8)\text{-}(9)$$

By Shephard's lemma,

$$l_i = \frac{\partial c_i(r, w)}{\partial w} = A_i (1-\alpha_i)\left(\frac{w}{r}\right)^{-\alpha_i}, \quad \forall\, i = 1, 2. \qquad (10)\text{-}(11)$$

$$K_i = k_i Q_i, \quad \forall\, i = 1, 2. \qquad (12)\text{-}(13)$$
$$L_i = l_i Q_i, \quad \forall\, i = 1, 2. \qquad (14)\text{-}(15)$$

Factor market clearing conditions:

$$K_1 + K_2 = \bar{K} \qquad (16)$$
$$L_1 + L_2 = \bar{L} \qquad (17)$$

FOREIGN GOODS MARKETS

Price equations:

$$P_i = \bar{P}_{Wi}, \quad \forall\, i = 1, 2. \qquad (18)\text{-}(19)$$

(Implied trade balance condition):

$$\bar{P}_{W1} E - \bar{P}_{W2} M = 0$$

VARIABLES IN THE MODEL

The 19 endogenous variables:

$X_1, X_2, Q_1, Q_2, K_1, K_2, L_1, L_2,$
$k_1, k_2, l_1, l_2, P_1, P_2, w, r, Y, E, M.$

The 4 exogenous variables:

$\bar{K}, \bar{L}, \bar{P}_{W1}, \bar{P}_{W2}.$

The 4 parameters

$\alpha_1, \alpha_2, a_1, a_2.$

APPENDIX G

Solutions for the closed economy

```
===========================the output file====================

 ///  MRS & p0/p1  ///

 (1) MRS=        0.480449773592573
 (2) p0/p1=      0.480449773592573


 ///  MRTS & w/r  ///

 (3) MRTS[0]=  0.400000000000000
 (4) MRTS[1]=  0.400000000000000
 (5) w/r=      0.400000000000000


 ///  MRT & p0/p1  ///

 (6) MRT=        0.480449773592572
 (7) p[0]/p[1]=  0.480449773592573


 ///  Income & expenditure  ///

 (8) Expendiure y=p0*c0+p1*c1 =        1.266870554312318
 (9) Factor income y2=r*kbar+w*lbar =  1.266870554312318


 ///  Factor markets  ///

 (10) k[0]=        0.080000000000000
 (11) k[1]=        0.720000000000000
 (12) k[0]+k[1]=  0.800000000000000
 (13) kbar=        0.800000000000000

 (14) l[0]=        1.800000000000000
 (15) l[1]=        0.200000000000000
 (16) l[0]+l[1]=  2.000000000000000
 (17) lbar=        2.000000000000000

 (18) rho_k= 2.7756e-17
 (19) rho_l= 5.5511e-17

 (20) k[0]/l[0]=  0.044444444444444
 (21) k[1]/l[1]=  3.600000000000000
 (22) kbar/lbar=  0.400000000000000
      NOTE: Sector 0 is relatively more labor-intensive !
      NOTE: The economy's relative endowment ratio kbar/lbar lies
            within the cone of diversification.
            Therefore, both goods are produced.

 (23) r=    0.791794096445199
 (24) w=    0.316717638578080
 (25) w/r=  0.400000000000000
 (26) r/w=  2.500000000000000


 ///  Commodity markets  ///
```

```
(27) Q[0]=  1.318421429194635
(28) Q[1]=  0.633435277156158

(29) x[0]=  1.318421429194635
(30) x[1]=  0.633435277156159

(31) p[0]/p[1]=   0.480449773592573

///  Welfare (utility) level  ///

(32) u_autarky =  0.913857014751499


///  #s of iterations  ///

(33) Iteration for general equilibrium loop: No.= 24

(34) The computational time:    0.000 seconds have passed.

================The end of the output file=============
```

APPENDIX H

Solutions for the small open economy

```
==========================the output file===================

 ///  MRS & p0/p1  ///

 (1) MRS=        0.276000000000000
 (2) q=          0.276000000000000
 (3) pw0/pw1=    0.276000000000000
 (4) p0/p1=      0.276000000000000


 ///  MRTS & w/r  ///

 (5) MRTS[0]=  0.200048985168461
 (6) MRTS[1]=  0.200048985168461
 (7) w/r=      0.200048985168461


 ///  MRT & p0/p1  ///

 (8) MRT=        0.276000000000000
 (9) p[0]/p[1]=  0.276000000000000


 ///  Income & expenditure  ///

 (10) Expendiure y=pw0*c0+pw1*c1 =     1.018406879737347
 (11) Factor income y2=r*kbar+w*lbar = 1.018406879737347


 ///  Factor markets  ///

 (12) k[0]=        0.035011021662904
 (13) k[1]=        0.764988978337096
 (14) k[0]+k[1]=   0.800000000000000
 (15) kbar=        0.800000000000000

 (16) l[0]=        1.575110189640746
 (17) l[1]=        0.424889810359254
 (18) l[0]+l[1]=   2.000000000000000
 (19) lbar=        2.000000000000000

 (20) rho_k= 6.2450e-17
 (21) rho_l= 0.0000e+00

 (22) k[0]/l[0]=  0.022227665018718
 (23) k[1]/l[1]=  1.800440866516147
 (24) kbar/lbar=  0.400000000000000
      NOTE: Sector 0 is relatively more labor-intensive !
      NOTE: The economy's relative endowment ratio kbar/lbar lies
            within the cone of diversification.
            Therefore, both goods are produced.

 (25) r=     0.848603118169956 (Determined by iteration, given an initial value)
 (26) w=     0.169762192600691
 (27) w/r=   0.200048985168461
 (28) r/w=   4.998775670658379


 ///  Commodity markets  ///
```

```
(29) Q[0]=  1.076466020052751
(30) Q[1]=  0.721302258202787

(31) x[0]=  1.844939999524179
(32) x[1]=  0.509203439868674

(33) p[0]/p[1]=    0.276000000000000
(34) q=            0.276000000000000
(35) pw[0]/pw[1]= 0.276000000000000


///  International trade  ///


(36) import= 7.6847e-01
(37) export= 2.1210e-01
     NOTE: Good 0 for import, and good 1 for export !

(38) import_value= 2.1210e-01
(39) export_value= 2.1210e-01
(40) trade_balance= -3.8858e-16


///  Welfare (utility) level  ///

(41) u_autarky =  0.913857014751499
(42) u_open    =  0.969252182927137
     NOTE: Utlity 'increased' !


///  #s of iterations  ///

(43) Iteration for general equilibrium loop: No.= 1

(44) The computational time:    0.000 seconds have passed.

================The end of the output file=============
```

APPENDIX I

Solutions for the small open economy with production tax on Q_2

```
==========================the output file===================

/// MRS & p0/p1 ///

(1) MRS=       0.480449773592573
(2) q=         0.480449773592573
(3) pw0/pw1=   0.480449773592573
(4) p0/p1=     1.681574207574005


/// MRTS & w/r ///

(5) MRTS[0]=  1.914895359814335
(6) MRTS[1]=  1.914895359814335
(7) w/r=      1.914895359814335


/// MRT & p0/p1 ///

(8) MRT=        1.681574207574006
(9) p[0]/p[1]=  1.681574207574005


/// Income & expenditure ///

(10) Total expendiure y=pw0*c0+pw1*c1 =         3.470880584079075
(11) Total value of production y=pw0*x0+pw1*x1 = 3.470880584079075
(12) Total income y1=r*kbar+w*lbar+transfer =   3.470880584079075
(13) Factor income y2=r*kbar+w*lbar =           2.827644572190750
(14) transfer =                                 0.643236011888325
(15) y2+transfer=                               3.470880584079075


/// Factor markets ///

(16) k[0]=        0.420851455958225
(17) k[1]=        0.379148544041775
(18) k[0]+k[1]=   0.800000000000000
(19) kbar=        0.800000000000000

(20) l[0]=        1.978000042775848
(21) l[1]=        0.021999957224152
(22) l[0]+l[1]=   2.000000000000000
(23) lbar=        2.000000000000000

(24) rho_k= 0.0000e+00
(25) rho_l= -8.3267e-17

(26) k[0]/l[0]=  0.212766151090482
(27) k[1]/l[1]= 17.234058238329023
(28) kbar/lbar=  0.400000000000000
     NOTE: Sector 0 is relatively more labor-intensive !
     NOTE: The economy's relative endowment ratio kbar/lbar lies
           within the cone of diversification.
           Therefore, both goods are produced.

(29) r=    0.610749976279174
(30) w=    1.169522295583705
(31) w/r=  1.914895359814335
(32) r/w=  0.522221746935017
```

```
///  Commodity markets  ///

(33) Q[0]=  1.694402370640438
(34) Q[1]=  0.285213867027113

(35) x[0]=  1.144020835056361
(36) x[1]=  0.549644551188015

(37) p[0]/p[1]=   1.681574207574005
(38) q=           0.480449773592573
(39) pw[0]/pw[1]= 0.480449773592573


///  International trade  ///


(40) import= 2.6443e-01
(41) export= 5.5038e-01
     NOTE: Good 1 for import, and good 0 for export !

(42) import_value=  8.3491e-01
(43) export_value=  8.3491e-01
(44) trade_balance= -1.9190e-17


///  Welfare (utility) level  ///

(45) u_autarky =  0.913857014751499
(46) u_p_tax   =  0.792972142281361
     NOTE: Utility 'lowered' !


///  Tax revenue & transfer  ///

(47) commodity_tax_revenue =  0.643236011888325
(48) total_tax_revenue=       0.643236011888325
(49) transfer =               0.643236011888325


///  Tax policy parameter  ///

(50) tau[0] =  0.000000000000000
(51) tau[1] =  2.500000000000000 (A given value)


///  #s of iterations  ///

(52) Iteration for general equilibrium loop: No.= 1

(53) The computational time:    0.000 seconds have passed.

================The end of the output flie=============
```

APPENDIX J

Solutions for the small open economy with consumption tax on C_1

```
==========================the output file===================

 ///  MRS & p0/p1  ///

 (1) MRS=        0.160149924530858
 (2) q=          0.160149924530858
 (3) pw0/pw1=    0.480449773592573
 (4) p0/p1=      0.480449773592573


 ///  MRTS & w/r  ///

 (5) MRTS[0]=  0.400000000000000
 (6) MRTS[1]=  0.400000000000000
 (7) w/r=      0.400000000000000


 ///  MRT & p0/p1  ///

 (8) MRT=          0.480449773592573
 (9) p[0]/p[1]=  0.480449773592573


 ///  Income & expenditure  ///

 (10) Total expendiure y=pw0*c0+pw1*(1+tau[0])*c1= 5.999999999999994
 (10-1) Total expendiure y=pw0*c0+pw1*c1        = 3.999999999999996
 (11) Total value of production y=pw0*x0+pw1*x1 =  3.999999999999996
 (12) Total income y1=r*kbar+w*lbar+transfer =     5.999999999999996
 (13) Factor income y2=r*kbar+w*lbar =             3.999999999999998
 (14) transfer =                                   1.999999999999998
 (15) y2+transfer=                                 5.999999999999996


 ///  Factor markets  ///

 (16) k[0]=         0.080000000000000
 (17) k[1]=         0.720000000000000
 (18) k[0]+k[1]=  0.800000000000000
 (19) kbar=         0.800000000000000

 (20) l[0]=         1.800000000000000
 (21) l[1]=         0.200000000000000
 (22) l[0]+l[1]=  2.000000000000000
 (23) lbar=         2.000000000000000

 (24) rho_k= -4.1633e-17
 (25) rho_l= -8.3267e-17

 (26) k[0]/l[0]=  0.044444444444444
 (27) k[1]/l[1]=  3.600000000000002
 (28) kbar/lbar=  0.400000000000000
      NOTE: Sector 0 is relatively more labor-intensive !
      NOTE: The economy's relative endowment ratio kbar/lbar lies
            within the cone of diversification.
            Therefore, both goods are produced.

 (29) r=    2.499999999999998
 (30) w=    1.000000000000000
```

```
(31) w/r=  0.400000000000000
(32) r/w=  2.499999999999999


///  Commodity markets  ///

(33) Q[0]=  1.318421429194635
(34) Q[1]=  0.633435277156158

(35) x[0]=  1.977632143791951
(36) x[1]=  0.316717638578079

(37) p[0]/p[1]=   0.480449773592573
(38) q=           0.160149924530858
(39) pw[0]/pw[1]= 0.480449773592573


///  International trade  ///


(40) import= 6.5921e-01
(41) export= 3.1672e-01
     NOTE: Good 1 for import, and good 0 for export !

(42) import_value=  1.0000e+00
(43) export_value=  1.0000e+00
(44) trade_balance= 6.9985e-16


///  Welfare (utility) level  ///

(45) u_autarky =  0.913857014751499
(46) u_c_tax   =  0.791423390201409
     NOTE: Utility 'lowered' !


///  Tax revenue & transfer  ///

(47) consumption_tax_revenue =  1.999999999999998
(48) total_tax_revenue=         1.999999999999998
(49) transfer =                 1.999999999999998


///  Tax policy parameter  ///

(50) tau[0] =  0.000000000000000
(51) tau[1] =  2.000000000000000 (A given value)


///  #s of iterations  ///

(52) Iteration for general equilibrium loop: No.= 32

(53) The computational time:    0.000 seconds have passed.


===============The end of the output file============
```