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Anti-duality for Cooperative Game Theory and Its Applications to Population Monotonicity

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We demonstrate that the notion of anti-duality, proposed by Oishi et al. (2015) and Oishi (2015), is highly useful for analyzing a new monotonic property for single-valued solutions for coalitional games with transferable utility (i.e. TU games). First, we propose a new monotonic property derived from the anti-dual of *population monotonicity*. This new property, referred to as *coalitional contribution monotonicity*, says that if the contribution of agents to a particular coalition to which they do not belong makes the coalitional worth for the agents in a new game, then they weakly gain in this game. Using the notion of anti-duality, we offer sufficient conditions under which *coalitional contribution monotonicity* is satisfied by a single-valued solution on the domain of convex games. Finally, we show that on the domain of all TU games, no single-valued solution satisfies *efficiency* and *coalitional contribution monotonicity*.

1 Introduction

Recently, Oishi et al. (2015) proposed an analytical framework for axiomatizations of solutions for coalitional games with transferable utility (TU games, for short). This framework is referred to as the "anti-duality approach". Using this framework, Oishi et al. (2015) provided new axiomatizations of the core (Gilles 1959) on the domain of balanced TU games, and of the Shapley value (Shapley 1953) on the domain of all TU games. The core and the Shapley value are main solutions for cooperative game theory, and they have many economic applications. Using the anti-duality approach, Oishi (2015) axiomatized some allocation rules for various economics problems. In these senses, the anti-duality approach is highly useful for characterizing both solutions for TU games and allocation rules for economics problems.

The purpose of this paper is to demonstrate that the anti-duality approach is useful for another game-theoretic analysis as well. For this purpose, we will focus on "population monotonicity" in cooperative game theory, and apply the notion of anti-duality to analyzing a new monotonic property that is the anti-dual of population monotonicity. Population monotonicity, originally proposed by Thomson (1983) in bargaining theory, says that if new agents arrive, the payoffs to agents that are present initially have to increase. Population monotonicity is known as an important property for axiomatic analysis in the situation where the number of agents is variable. For the detail of population monotonicity, see Thomson and Lensberg (1989).

Our results are summarized as follows: First, we derive a new monotonic property as the anti-dual of population monotonicity. A new monotonicity we propose says that if the contribution of agents to a particular coalition to which they do not belong makes the coalitional worth for the agents in a new game, then they weakly gain in this game. We refer to it as "coalitional contribution monotonicity". Hokari and Gellekom (2002) provided us with sufficient conditions for a single-valued solution to be population monotonic on the domain of convex games. Using the anti-duality approach, we derive suffcient conditions for a single-valued solution to be coalitional contribution

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monotonic on the domain of convex games from Hokari and Gellekom's conditions. Finally, we show that on the domain of all TU games no-single valued solution satisfies efficiency and coalitional contribution monotonicity. We derive the anti-dual result of this impossibility theorem: On the domain of all TU gems, no-single valued solution satisfies efficiency and population monotonicity.

The rest of this paper is organized as follows. Following Oishi et al. (2015) and Oishi (2015), in Section 2, we explain the notion of anti-duality for cooperative game theory. In Section 3, we introduce coalitional contribution monotonicity. On the domain of convex games, we offer sufficient conditions under which coalitional contribution monotonicity is satisfied by a singlevalued solution. In Section 4, we show the impossibility theorem concerning coalitional contribution monotonicity.

2 Preliminaries

Following Oishi (2015) and Oishi et al. (2015), we explain the notion of anti-duality for solutions and axioms for cooperative game theory. There is a universe of potential agents, denoted $\mathcal{I} \subseteq \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.[1] Let $\mathcal{N}$ be the class of non-empty and finite subsets of $\mathcal{I}$, and $N \in \mathcal{N}$. **A coalitional game with transferable utility for** $\boldsymbol{N}$ (a **TU game for** $\boldsymbol{N}$, for short) is a function $v: 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. For all $S \in 2^N$, $v(S)$ represents what coalition S can achieve on its own. Let $\boldsymbol{\mathcal{V}^N}$ be the **class of TU games for** $\boldsymbol{N}$, and $\mathcal{V} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{V}^N$.

Given a TU game v for N and $N' \subset N$, the **subgame of** $\boldsymbol{v}$ **relative to** $\boldsymbol{N'}$, denoted $v|_{N'}$, is defined by setting, for all $S \in 2^{N'}$, $v|_{N'}(S) \equiv v(S)$. A TU game v for N is **convex** if for all $i \in N$ and all $S; T \subseteq N \backslash \{i\}$, $S \subseteq T$ implies $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. Let $\boldsymbol{\mathcal{V}^N_{vex}}$ be the **class of convex games for** $\boldsymbol{N}$, and $\mathcal{V}_{vex} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{V}^N_{vex}$. A TU game v for N is **balanced** if for all non-negative function $\delta : 2^N \to \mathbb{R}_+$ such that for all $i \in N$, $\sum_{S \ni i} \delta(S) = 1$, $v(N) \geq \sum_{S \in 2^N} \delta(S) v(S)$. A *convex* game is *balanced.*

Let $\mathbb{R}^N$ denote the Cartesian product of $|N|$ copies of $\mathbb{R}$, indexed by the members of N. A **payoff vector** for N is an element x of $\mathbb{R}^N$. For all $x \in \mathbb{R}^N$ and all $S \in 2^N$, let $x_S = (x_i)_{i \in S}$. A **solution**, denoted $\boldsymbol{\varphi}$, is a mapping, defined on some domain of games, that associates with each game in the domain a nonempty set of payoff vectors. A solution is **single-valued** if it associates with each game in its domain a unique payoff vector.

Given a game v for N, the **dual of** $\boldsymbol{v}$, denoted $\boldsymbol{v^d}$, is defned by setting, for all $S \subseteq N$,

$$v^d(S) \equiv v(N) - v(N \backslash S).$$

The number $v^d(S)$ is the amount that the complementary coalition $N \backslash S$ cannot prevent S from obtaining.

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^d \in \mathcal{V}$. Given a solution φ on $\mathcal{V}$, the **dual of** $\boldsymbol{\varphi}$, denoted $\boldsymbol{\varphi^d}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^d(v) \equiv \varphi(v^d).$$

A solution φ on $\mathcal{V}$ is **self-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^d(v)$.

An **axiom** is a desirable property of solutions. **Two axioms are dual of each other** if whenever a solution satisfies one of them, the dual of this solution satisfies the other. **An axiom is self-dual** if it is its own dual.

Given a game v for N, the **anti-dual of** $\boldsymbol{v}$, denoted $\boldsymbol{v^{ad}}$, is defined by setting, for all $S \subseteq N$,

$$v^{ad}(S) \equiv -v^d(S).^{2}$$

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^{ad} \in \mathcal{V}$. The class of balanced games and the class of convex games satisfy this property. Given a solution φ on $\mathcal{V}$, the **anti-dual** of $\boldsymbol{\varphi}$, denoted $\boldsymbol{\varphi^{ad}}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^{ad}(v) \equiv -\varphi(v^{ad}).$$

A solution φ on $\mathcal{V}$ is **self-anti-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^{ad}(v)$. **Two axioms are anti-dual of each other** if whenever a solution satisfies one of them, the anti-dual of this solution satisfies the other. **An axiom is self-anti-dual** if it is its own anti-dual.

Finally, we introduce well-known solutions for coalitional games. The **core** (Gillies 1959) is defined as follows: for all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N$,

$$C(v) \equiv \Big\{x \in \mathbb{R}^N \;\Big|\; \sum_{i \in N} x_i = v(N) \text{ and for all } S \subseteq N, \sum_{i \in S} x_i \geq v(S)\Big\}.$$

The core of v for N is not empty if and only if the game v is *balanced.*

The **Shapley value** (Shapley 1953) is defined as follows: for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $i \in N$,

$$Sh_i(v) \equiv \sum_{\substack{S \subseteq N \\ S \not\ni i}} \frac{|S|!(|N|-|S|-1)!}{|N|!}\big[v(S \cup \{i\}) - v(S)\big].$$

On the domain of *convex* games, the core is never empty, and the Shapley value is a selection from the core.

Given $N \in \mathcal{N}$ and $v \in \mathcal{V}^N$, let $I(v)$ be the set of vectors $x \in \mathbb{R}^N$ such that for all $i \in N$, $x_i \geq v(\{i\})$, and $\sum_N x_i = v(N)$ For all $x \in I(v)$, let $e(v,x) \in \mathbb{R}^{2^N}$ be defined by setting, for all $S \subseteq N$, $e_S(v,x) \equiv v(S) - \sum_S x_i$. For all $z \in \mathbb{R}^{2^N}$, $\theta(z) \in \mathbb{R}^{2^N}$ is defined by rearranging the coordinates of z in non-increasing order. For all $z \in \mathbb{R}^{2^N}$, ***z* is lexicographically smaller than *z'*** if $\theta_1(z) < \theta_1(z')$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) < \theta_2(z')]$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) = \theta_2(z')$ and $\theta_3(z) < \theta_3(z')]$, and so on. The **nucleolus** (Schmeidler 1969) is defined as follows:

$$Nu(v) \equiv \left\{x \in I(v) \;\middle|\; \begin{array}{c}\text{For all } y \in I(v)\backslash\{x\},\ e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,y)\end{array}\right\}.$$

The nucleolus is a *single-valued* solution. On the domain of *convex* games, the nucleolus is a selection form the core.

3 Anti-duality approach to population monotonicity

In this section, we deal with a *single-valued* solution, denoted φ, on some domain of games. We write $x = \varphi(v)$ instead of $\{x\} = \varphi(v)$.

Let us consider the well-known monotonic property in cooperative game theory, *population monotonicity* (Thomson 1983; Sprumont 1990). This monotonic property says that for all game $v \in \mathcal{V}^N$ and all subgames $v|_{N'} \in \mathcal{V}^{N'}$, if agents play in $v|_{N'}$, then the payoffs to the agents in v have to increase. Formally, this property is as follows:

Population monotonicity: For all N, $N' \in \mathcal{N}$ such that $N' \subset N$, all $v \in \mathcal{V}^N$, and all $i \in N'$, $\varphi_i(v|_{N'}) \leq \varphi_i(v)$.

Let us introduce the following monotonic property. An interpretation of this property is as follows: We start with some game v for $N \in \mathcal{N}$. Next, we consider the game v^c played by $N' \subset N$. It is meant to describe the situation where the worth of each coalition $S \subseteq N'$ is equal to the contribution of S to $N \backslash N'$ in v. This property says that in the game v^c, the payoffs to the members of N' have to be at least as large as in v. Note that v^c is not $v|_{N'}$.

Coalitional contribution monotonicity: For all N, $N' \in \mathcal{N}$ such that $N' \subset N$, all $v \in \mathcal{V}^N$, all $v' \in \mathcal{V}^{N'}$ such that for all $S \subseteq N'$ $v^c(S) \equiv v(S \cup (N \backslash N')) - v(N \backslash N')$, and all $i \in N'$,

$$\varphi_i(v^c) \geq \varphi_i(v).$$

The following claim says that v^c is *convex* on the domain of convex games.

Claim 1 *For all N, $N' \in \mathcal{N}$ such that $N' \subset N$, all $v \in \mathcal{V}^N_{vex}$, and all $S \subseteq N'$,*

$$v^c(S) \equiv v(S \cup (N \backslash N')) - v(N \backslash N').$$

Then,

$$v^c \in \mathcal{V}^{N'}_{vex}.$$

Proof. A game is *convex* if for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ For all $S, T \subseteq N'$,

$v^c(S)+v^c(T)$

$=v(S\cup(N\backslash N'))-v(N\backslash N')+v(T\cup(N\backslash N'))-v(N\backslash N')$

$\leq v\left((S\cup(N\backslash N'))\cup(T\cup(N\backslash N'))\right)-v(N\backslash N')$

$\quad+v\left((S\cup(N\backslash N'))\cap(T\cup(N\backslash N'))\right)-v(N\backslash N')$

$=v\left((S\cup T)\cup(N\backslash N')\right)-v(N\backslash N')$

$\quad+v\left((S\cap T)\cup(N\backslash N')\right)-v(N\backslash N')$

$=v^c(S\cup T)+v^c(S\cap T)$,

the desired conclusion. ■

The class of convex games is closed under the *anti-duality* operator, but *not under the duality* operator.[3] Using the anti-duality operator, we obtain the following result:

Proposition 1 *On the domain of convex games, population monotonicity and coalitional contribution monotonicity are anti-dual of each other.*

Proof. Let φ be a *population monotonic* solution on $\mathcal{V}^N_{vex}$. Let $v\in\mathcal{V}^N_{vex}$, $x\equiv\varphi^{ad}_{N'}(v)$ and $y\equiv\varphi^{ad}(v|_{N'})$.[4] For all $S\subseteq N'$, $w(S)\equiv v^{ad}|_{N'}(S)$. By the definition of φ^{ad}, $-x\equiv\varphi_{N'}(v^{ad})$ and $-y\equiv\varphi(w)$. Note that $v^{ad}\in\mathcal{V}^N_{vex}$ and $w\in\mathcal{V}^{N'}_{vex}$. Since φ is *population monotonic*, for all $i\in N'$, $\varphi_i(w)\leq\varphi_i(v^{ad})$. By the definition of φ^{ad}, for all $i\in N'$, $\varphi^{ad}_i(w^{ad})\geq\varphi^{ad}_i(v)$.

For all $S\subseteq N'$,

$$\begin{aligned} w^{ad}(S) &= -w(N')+w(N'\backslash S)\\ &= v^d|_{N'}(N')-v^d|_{N'}(N'\backslash S)\\ &= v(N)-v(N\backslash N')-v(N)+v(N\backslash(N'\backslash S))\\ &= v(S\cup(N\backslash N'))-v(N\backslash N'), \end{aligned}$$

the desired conclusion. ■

The followings are well-known properties of single-valued solutions (e.g., Peleg and Sudhölter 2003):

Efficiency:For all $N\in\mathcal{N}$, and all $v\in\mathcal{V}^N$, $\sum_N\varphi_i(v)=v(N)$.

Individual rationality: For all $N\in\mathcal{N}$, all $v\in\mathcal{V}^N$, and all $i\in N$, $\varphi_i(v)\geq v(\{i\})$.

Reasonableness: For all $N\in\mathcal{N}$, all $v\in\mathcal{V}^N$, and all $i\in N$, $\varphi_i(v)\leq v(N)-v(N\backslash\{i\})$.[5]

Lemma 1 *On the domain of convex games, (i) efficiency is self-anti-dual, and (ii) individual rationality and reasonableness are anti-dual of each other.*

Proof. Immediately from the definition of anti-duality. ■

For all N, $N'\subset N$ such that $N'\subset N$, and all $v\in\mathcal{V}^N$, the **self-reduced game of v relative to φ and N'**, denoted $r^{\varphi}_{N'}(v)$, is defined by setting, for all $S\subseteq N'$,

$$r^{\varphi}_{N'}(v)(S)\equiv\begin{cases} v(N)-\sum_{N\backslash N'}\varphi_i(v) & \text{if } S=N'\\ v\left(S\cup(N\backslash N')\right)-\sum_{N\backslash N'}\varphi_i(v\,|_{S\cup(N\backslash N')}) & \text{if } S\neq N',\emptyset,\\ 0 & \text{if } S=\emptyset. \end{cases}$$

Consider the following property associated with *self-reduced games*:

Self consistency: For all N, $N'\in\mathcal{N}$ such that $N'\subset N$ and all $v\in\mathcal{V}^N$, we have $r^{\varphi}_{N'}(v)\in\mathcal{V}^{N'}$ and for all $i\in N'$, $\varphi_i(r^{\varphi}_{N'}(v))=\varphi_i(v)$.

Self consistency (Hart and Mas-Colell 1989) reguires that the outcome a solution chooses for each game in $\mathcal{V}^N$ should be equal to the outcome chosen by the solution for the *self-reduced game relative to φ and N'*.[6]

On the domain of all TU games, the Shapley value is *self consistent*. However, on the domain of *convex* games, it is not, since $\mathcal{V}_{vex}$ is not closed under the *self-reduction operator* for this solution.

The following notion is a weaker notion:

Bilateral self-consistency: For all N, $N'\in\mathcal{N}$ such that $N'\subset N$ with $|N'|$=2, and all $v\in\mathcal{V}^N$, we have $r^{\varphi}_{N'}(v)\in\mathcal{V}^{N'}$ and for all $i\in N'$, $\varphi_i(r^{\varphi}_{N'}(v))=\varphi_i(v)$. On the domain of *convex* games, the Shapley value is *bilaterally self-consistent.*

Let us consider the following alternative notion of consistency, introduced by Oishi et al. (2015).

Transfer agreement consistency (Oishi et al. 2015): For all N, $N'\in\mathcal{N}$ such that $N'\subset N$, and all $v\in\mathcal{V}^N$, if for all $S\subseteq N'$,

$\tilde{r}^{\varphi}_{N'}(v)(S)\equiv$

$$\begin{cases} v(N) - \sum_{N\setminus N'} \varphi_i(v) & \text{if } S = N', \\ v(S) + \sum_{N\setminus N'} \varphi_i(v^{N\setminus S}) - \sum_{N\setminus N'} \varphi_i(v) & \text{if } S \neq N', \emptyset, \\ 0 & \text{if } S = \emptyset, \end{cases}$$

where $v^{N\setminus S}$ is the game for $N\setminus S$ defned by setting, for all $T \subseteq N\setminus S, v^{N\setminus S}(T) \equiv v(S\cup T) - v(S)$, we have $\tilde{r}^{\varphi}_{N'}(v) \in \mathcal{V}^{N'}$ and for all $i \in N'$, $\varphi_i(\tilde{r}^{\varphi}_{N'}(v)) = \varphi_i(v)$.

The following notion weakens *transfer agreement consistency* by limiting its application to subpopulation of two agents:

Bilateral transfer-agreement consistency: For all N , $N' \in \mathcal{N}$ such that $N' \subset N$ with $|N'|$ = 2, and all $v \in \mathcal{V}^N$, we have $\tilde{r}^{\varphi}_{N'}(v) \in \mathcal{V}^{N'}$ and for all $i \in N'$ $\varphi_i(\tilde{r}^{\varphi}_{N'}(v)) = \varphi_i(v)$.

Bilateral transfer-agreement consistency can be described as follows: Let $N \in \mathcal{N}$ and $N' = \{i, j\}$ with $i \neq j$. The games $v^{N\setminus\{i\}}$ and $v^{N\setminus\{j\}}$ are given by setting,

for all $T \subseteq N\setminus\{i\}$, $v^{N\setminus\{i\}}(T) = v(\{i\}\cup T) - v(\{i\})$;

for all $T \subseteq N\setminus\{j\}$, $v^{N\setminus\{j\}}(T) = v(\{j\}\cup T) - v(\{j\})$.

Then, $\tilde{r}^{\varphi}_{N'}$ is given by

$$\tilde{r}^{\varphi}_{N'}(v)(\{i\}) = v(\{i\}) + \sum_{N\setminus N'} \left[\varphi_k(v^{N\setminus\{i\}}) - \varphi_k(v)\right],$$

$$\tilde{r}^{\varphi}_{N'}(v)(\{j\}) = v(\{j\}) + \sum_{N\setminus N'} \left[\varphi_k(v^{N\setminus\{j\}}) - \varphi_k(v)\right],$$

$$\tilde{r}^{\varphi}_{N'}(v)(N') = v(N) - \sum_{N\setminus N'} \varphi_k(v).$$

The scenario underlying the reduced game $\tilde{r}^{\varphi}_{N'}$ is as follows. The scenario here is proposed by Oishi et al. (2015).

Imagine that agent i announces that he will cooperate with anybody if he obtains $v(\{i\})$. If some agents, who form a coalition $T \subseteq N\setminus\{i\}$, cooperate with agent i, the coalition $\{i\}\cup T$ obtains $v(\{i\}\cup T)$. Since the reward of agent i for his cooperation is $v(\{i\})$, the coalition T obtains the remainder $v(\{i\}\cup T) - v(\{i\})$. Thus, the agents except for agent i play $v^{N\setminus\{i\}}$ and obtain $\varphi(v^{N\setminus\{i\}})$. If agent i does not make this announcement, the agents except for agent i obtain $\varphi(v)$. If agents i and $k \in N\setminus N'$ agree that the difference $\varphi_k(v^{N\setminus\{i\}}) - \varphi_k(v)$ should be transferred from agent k to agent i, then agent i obtains $\tilde{r}^{\varphi}_{N'}(v)(\{i\})$ as defined above. The worth $\tilde{r}^{\varphi}_{N'}(v)(\{j\})$ can be interpreted in the same manner.

Bilateral transfer-agreement consistency reguires that what agents i and j get should be unchanged even if such an agreement between agents i and $k \in N\setminus N'$ or between agents j and $k \in N\setminus N'$ is taken place.

Proposition 2 *On the domain of convex games, bilateral self-consistency and bilateral transfer-agreement consistency are anti-dual of each other.*

Proof. Let φ be a *single-valued* solution on $\mathcal{V}^N_{vex}$ that satisfies *bilateral self-consistency*. Let $N \in \mathcal{N}$, v be a *convex* game for N , and $x \equiv \varphi^{ad}(v)$. By the definition of φ^{ad} , $x = -\varphi(-v^d)$. Let $N' \subset N$ with $|N'|$ = 2, and $w \in \mathbb{R}^{2^{N'}}$ be such that for all $S \subseteq N'$,

$w(S)$=

$$\begin{cases} -v^d(N) + \sum_{i\in N\setminus N'} x_i & \text{if } S = N', \\ -v^d\left(S\cup(N\setminus N')\right) - \sum_{i\in N\setminus N'} \varphi_i\left(-v^d|_{S\cup(N\setminus N')}\right) & \text{if } S \neq N', \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Since φ satisfes *bilateral self-consistency*, $w \in \mathcal{V}^{N'}_{vex}$ and $x_{N'} = -\varphi(w)$. Again by the defnition of φ^{ad}, $x_{N'} = \varphi^{ad}(-w^d)$.

First, we have

$$-w^d(N') = v^d(N) - \sum_{i\in N\setminus N'} x_i = v(N) - \sum_{i\in N\setminus N'} \varphi^{ad}_i(v).$$

Next, for all $S \subset N'$ with $S \neq \emptyset$,

$$\begin{aligned} w(S) &= -v^d\left(S\cup(N\setminus N')\right) - \sum_{i\in N\setminus N'} \varphi_i\left(-v^d|_{S\cup(N\setminus N')}\right) \\ &= -v(N) + v\left(N'\setminus S\right) - \sum_{i\in N\setminus N'} \varphi_i\left(-v^d|_{S\cup(N\setminus N')}\right). \end{aligned}$$

Thus, for all $S \subset N'$ with $S \neq \emptyset$, we have

$$\begin{aligned} -w^d(S) &= -w(N') + w(N'\setminus S) \\ &= v(N) - \sum_{i\in N\setminus N'} x_i - v(N) + v\left(N'\setminus(N'\setminus S)\right) \\ &\quad - \sum_{i\in N\setminus N'} \varphi_i\left(-v^d|_{(N'\setminus S)\cup(N\setminus N')}\right) \\ &= -\sum_{i\in N\setminus N'} x_i + v(S) - \sum_{i\in N\setminus N'} \varphi_i\left(-v^d|_{N\setminus S}\right), \end{aligned}$$

so that

$$-w^d(S) = v(S) + \sum_{i\in N\setminus N'} \varphi^{ad}_i\left(-\left(-v^d|_{N\setminus S}\right)^d\right) - \sum_{i\in N\setminus N'} \varphi^{ad}_i(v).$$

Note that for all $T \subseteq N \backslash S$,

$$-v^d|_{N\backslash S}(T) = -v^d(T) = -v(N) + v(N\backslash T).$$

Thus, for all $T \subseteq N \backslash S$,

$$\begin{aligned} -\left(-v^d|_{N\backslash S}\right)^d(T) &= -(-v^d|_{N\backslash S})(N\backslash S) + (-v^d|_{N\backslash S}) \\ &\quad ((N\backslash S)\backslash T) \\ &= v(N) - v(N\backslash(N\backslash S)) - v(N) + \\ &\quad v\left(N\backslash((N\backslash S)\backslash T)\right) \\ &= v(S \cup T) - v(S) \\ &= v^{N\backslash S}(T), \end{aligned}$$

the desired conclusion. ■

Hokari and Gellekom (2002) identified the following sufficient conditions for a single-valued solution to be *population monotonic* on the domain of *convex* games.

Proposition A *(Hokari and Gellekom 2002) On the domain of convex games, if a single-valued solution is efficient, individual rational, and bilaterally selfconsistent, then it is population monotonic.*

Corollary 1 *On the domain of convex games, the Shapley value is population monotonic (Sprumont 1990).*

Using the *anti-duality* operator, we derive sufficient conditions for a singlevalued solution to be *coalitional contribution monotonic* on the domain of *convex* games from Hokari and Gellekom's conditions.

Proposition 3 *(Anti-dual of Proposition A) On the domain of convex games, if a single-valued solution is efficient, reasonable, and bilaterally transfer-agreement consistent, then it is coalitional contribution monotonic.*

Proof. *By Propositions 1 and 2, and Lemma 1.* ■

On the domain of *convex* games $\mathcal{V}_{vex}$, the Shapley value is *self-anti-dual* (Oishi and Nakayama 2009, Theorem 2). On $\mathcal{V}_{vex}$, it is *efficient, reasonable*, and *bilaterally transfer-agreement consistent*. Thus, the conditions stated in Proposition 3 are satisfied by the Shapley value.

4 Further discussion: Impossibility results

Finally, let us consider an application of *coalitional contribution monotonicity.*

One may wonder whether there is a solution satisfying *efficiency* and *coalitional contribution monotonicity* on the domain of all TU games. Unfortunately, we provide a negative answer for this question:

Theorem 1 *On the domain of all TU games, no single-valued solution satisfies efficiency and coalitional contribution monotonicity.*

Proof. Let $\mathcal{V}^N$ be the class of all TU games for N. Let φ be a solution on $\mathcal{V}^N$. Suppose that it satisfies the two properties. For all $N, N' \in \mathcal{N}$ such that $N' \subset N$, all $v \in \mathcal{V}^N$ and all $v^c \in \mathcal{V}^{N'}$ such that for all $S \subseteq N'$, $v^c(S) \equiv v(S \cup (N\backslash N')) - v(N\backslash N')$. By *coalitional contribution monotonicity,* for all $i \in N'$,

$$\sum_{N'} \varphi_i(v^c) \geq \sum_{N'} \varphi_i(v).$$

By *efficiency,*

$$\sum_{N'} \varphi_i(v^c) = v^c(N') = v(N) - v(N\backslash N'),$$

and

$$\sum_{N} \varphi_i(v) = v(N).$$

Thus, for all N, $N' \in \mathcal{N}$ such that $N' \subset N$,

$$\sum_{N} \varphi_i(v) - v(N\backslash N') \geq \sum_{N'} \varphi_i(v) \ ,$$

which implies $\sum_{N\backslash N'} \varphi_i(v) \geq v(N\backslash N')$. Therefore, for all $v \in \mathcal{V}^N$, $\varphi(v) \in C(v)$. However, for some *non-balanced game* $v \in \mathcal{V}^N$, $\varphi(v) \notin C(v)$, a contradiction. ■

Note that on the domain of all TU games, *population monotonicity* and *coalitional contribution monotonicity* are *anti-dual* of each other.[7] On this domain, *efficiency* is *self-anti-dual*. We derive the following impossibility theorem by applying the notion of *anti-duality* to Theorem 1.

Theorem 2 *(Anti-dual of Theorem 1) On the domain of all TU games, no single-valued solution* *satisfies efficiency and population monotonicity.*

Although Sprumont (1990) showed Theorem 2 without using the *antiduality* approach, we obtain the same result using this approach.

Corollary 2 *On the domain of convex games, if a single-valued solution is efficient and population monotonic, then it is a selection from the core.*

On the domain of *convex* games, the Shapley value is *efficient* and *population monotonic*. So is the Dutta-Ray solution (Dutta and Ray 1989). On this domain, these solutions are selections from the core. Although the nucleolus is a selection from the core, it is not *population monotonic* on the domain (Sönmez 1994). Therefore, the necessary condition stated in Corollary 2 is not sufficient for a single-valued solution to be *efficient* and *population monotonic.*

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Notes

1. We use ⊆ for weak set inclusion, and ⊂ for strict set inclusion.
2. The notion of anti-dual games is initially introduced by Oishi and Nakayama (2009).
3. Several important classes of TU games (such as balanced TU games and convex TU games) are not closed under the *duality* operator. For this reason, applicability of the duality approach to axiomatization of solutions is very limited. On the other hand, both the class of balanced TU games and that of convex TU games are closed under the *anti-duality* operator. Therefore, the *anti-duality* approach is advantageous to axiomatization of solutions. For the detail, see Oishi et al. (2015).
4. Note that $\varphi_{N'}^{ad}(v) \equiv \left(\varphi_i^{ad}(v)\right)_{i\in N'}$.
5. On the domain of convex games, it is "reason ableness from above" (Milnor 1952).
6. *Self-reduced game* and *self-reduced consistency* are usually called "HM-reduced game" and "HM-consistency", respectively. We use the terminology introduced by Thomson (1996).
7. The proof is the same as in Proposition 1.

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