Bosonization of Fermion Interaction

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1. Model

Many body problems have been extensively discussed since the 1950s and great progress has been made.[1] This progress was possible mainly because Feynman[2] invented an efficient diagrammatic method. Recently, however, there have appeared difficult problems such as strong interactions and large amplitude phenomena which can not be solved by straightforward application of Feynman diagrams.

Bosonization[1] is one of the methods available to approach those difficult problems. The method enables one to sum up a series of diagrams effectively so that even a non-perturbative effect can be taken into account. So far, the Bosonization method has been successfully used, especially in nuclear physics[3] and in low temperature physics.[4] For simple Fermion interactions the Bosonization can be achieved by Hubbard-Stratonovich[5] transformation through the path integral. Up to now, however, this transformation has been applied only to one kind of Boson. In the present paper two kinds of Bosons are introduced in the Bosonization.

Our model is given by

$$H_{\text{int}}=\frac{1}{2}\sum\int d^3\mathbf{x}d^3\mathbf{x}'\,\psi_\alpha{}^+(\mathbf{x})\,\psi_\beta{}^+(\mathbf{x}')v(\mathbf{x},\mathbf{x}')\,\psi_\beta(\mathbf{x}')\,\psi_\alpha(\mathbf{x}) \tag{1}$$

in the standard notation. The corresponding partition function in Euclidean formalism is then

$$Z=\int D\psi^+D\psi\exp(-A) \tag{2}$$

where the integral is the path integral over Grassmann numbers ψ^+, ψ and A is the action defined by

$$A=A_0+A_{\text{int}} \tag{3}$$

with the bare term

$$A_0=\psi^+(1)(\partial_\tau+\xi)_{12}\,\psi(2) \tag{4}$$

and the interaction

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$$A_{\rm int}=\frac{1}{2}\psi^{+}(1)\psi^{+}(2)v_{12}\psi(2)\psi(1). \qquad (5)$$

In these formulae the number signifies space, time (imaginary) and spin. From now on repeated numbers should be summed unless otherwise stated. The two-body interaction is defined by

$$v_{12}=v(12)=v(\mathbf{x}_1,\mathbf{x}_2)\,\delta(\tau_1-\tau_2)\,\delta(\alpha_1,\alpha_2). \qquad (6)$$

2. Hubbard-Stratonovich transformation

The original transformation[5] can be cast into Gaussiann integrals over Grassmann numbers.[4] For the present model in Eq. (2) it is

$$\int D\phi\,\exp(-A^{\rm H}_{\rm int}[\phi])=c\,\exp(-A_{\rm int}) \qquad (7)$$

where c is an unimportant factor and

$$A^{\rm H}_{\rm int}[\phi]=\frac{1}{2v(12)}(\phi v^{-1}+i\hat{n})_1 v_{12}(v^{-1}\phi+i\hat{n})_2+A_{\rm int} \qquad (8)$$

with $\hat{n}(1)=\psi^{+}(1)\psi(1)$. If the path integral over Grassmann numbers are performed, a purely Bosonic theory emerges,

$$Z=\int D\phi\,\exp(-A_{\rm eff}[\phi]) \qquad (9)$$

where the effective action is given by

$$A_{\rm eff}[\phi]=-{\rm tr}\,\ln(\partial_\tau+\xi+i\phi)+\frac{1}{2}\phi\,v^{-1}\,\phi\,. \qquad (10)$$

The new Boson describes a density fluctuation and is convenient to study the bubble diagram. This diagram is important for screening phenomena in the high density region.

The transformation can be extended for a bilocal Bosonic field[3] which depicts particle-hole pairs and corresponds to the ladder diagram. The formula extended is

$$\int D\rho\,\exp(-A^{\rm F}_{\rm int}[\rho])=c'\,\exp(-A_{\rm int}) \qquad (11)$$

where the auxiliary field ρ is hermitian and

$$A^{\rm F}_{\rm int}[\rho]=\frac{1}{2v(12)}|\,\rho(12)\,-v(12)\,\hat{\rho}(21)|^2+A_{\rm int} \tag{12}$$

with $\hat{\rho}(21)=\psi^+(2)\psi(1)$. Again, the path integral over Grassmann numbers makes the theory Bosonic,

$$Z=\int D\rho\,\exp(-A_{\rm eff}[\rho]) \tag{13}$$

where

$$A_{\rm eff}[\rho]=-{\rm tr}\,\ln(\partial_\tau+\xi-\rho)+\frac{1}{2v(12)}|\,\rho(12)|^2. \tag{14}$$

3. Extension

The two transformations in Section 2 can be combined to establish a single theory. The combination is useful because the first transformation is important in high density while the second is crucial in low density. The combined theory then would have wider applicability than either of the two transformations.

The combination of the transformations is easily achieved by making a product of Eqs. (11) and (7) after each action in Eqs. (12) and (8) is multiplied by a and b respectively. The factors must of course satisfy

$$a+b=1 \tag{15}$$

to ensure that the original theory in Eq. (2) is retained. The factors may be chosen to improve perturbative results. The partition function is now

$$Z=\int D\psi^+D\psi\,D\phi\,D\rho\,\exp(-A^{\rm H+F}) \tag{16}$$

where

$$A^{\rm H+F}=\psi^+(1)(\partial_\tau+\xi+{\rm i}\,\phi\,-\rho)_{12}\,\psi(2)+\frac{1}{2}\phi(1)(bv)^{-1}{}_{12}\,\phi(2)+\frac{1}{2}\frac{1}{av(12)}|\,\rho(12)|^2. \tag{17}$$

To reach the above equation the factors a and b were absorbed into the Bosons, and an irrelevant front factor for the partition function was ignored. As in Section 2 Grassmann numbers can be integrated out to produce the Bosonic theory described by ϕ and ρ ,

$$Z=\int D\phi\,D\rho\,\exp(-A_{\rm eff}[\phi,\rho]) \tag{18}$$

where the effective action is now

$$A_{\rm eff}[\phi,\rho]=\text{-tr}\ \ln(\partial_\tau+\xi+\mathrm{i}\phi-\rho)+\frac{1}{2}\phi(1)(bv)^{-1}{}_{12}\phi(2)+\frac{1}{2}\frac{1}{av(12)}|\rho(12)|^2. \tag{19}$$

4. One-loop calculation

To gain an insight into the effective theory in Eq. (18) the one-loop evaluation is presented here. The stationary values of the fields satisfy

$$\begin{cases}\mathrm{i}\phi_0(1)=-bv(12)g_{22}{}^{(0)}\\ \rho_0(12)=-av(12)g_{12}{}^{(0)}\end{cases} \tag{20}$$

where

$$g^{(0)}=(\partial_\tau+\xi+\mathrm{i}\phi_0-\rho_0)^{-1}. \tag{21}$$

The fluctuation of the fields from these stationary values is controlled in the lowest order by the action,

$$\begin{aligned}A_{\rm eff}{}^{(2)}[\delta\phi,\delta\rho]=&\frac{1}{2}\delta\rho(12)I^{-1}{}_{12;34}\delta\rho(34)\\ &-\mathrm{i}g_{41}{}^{(0)}g_{24}{}^{(0)}\delta\rho(12)\delta\phi(4)+\frac{1}{2}\delta\phi(1)d^{-1}{}_{12}\delta\phi(2)\end{aligned} \tag{22}$$

where the ladder sum I is defined by

$$I^{-1}{}_{12;34}=\frac{1}{av(12)}\delta(2,3)\delta(1,4)+g_{41}{}^{(0)}g_{23}{}^{(0)} \tag{23}$$

and the bubble sum d satisfies

$$d^{-1}{}_{12}=(bv)^{-1}{}_{12}-g_{12}{}^{(0)}g_{21}{}^{(0)}. \tag{24}$$

Since both fields are coupled to each other it is better to introduce a field doublet

$$\delta\Phi(12)=\begin{pmatrix}\delta\phi(1)\,\delta(1,2)\\ \delta\rho(12)\end{pmatrix}. \tag{25}$$

Then Eq. (22) is now

$$A_{\rm eff}{}^{(2)}[\delta\phi,\delta\rho]=\frac{1}{2}\delta\Phi^{t}(12)K^{-1}{}_{12;34}\delta\Phi(34) \tag{26}$$

where the propagator K is defined by

$$K^{-1}{}_{12;34} = \begin{pmatrix} I^{-1}{}_{12;34} & -\mathrm{i}g_{41}{}^{(0)}g_{23}{}^{(0)} \\ -\mathrm{i}g_{23}{}^{(0)}g_{41}{}^{(0)} & d^{-1}{}_{13} \end{pmatrix}. \tag{27}$$

Therefore, the energy up to the one-loop order is given by

$$E_{0+1} = -\mathrm{tr}\ \ln(\partial_\tau + \xi + \mathrm{i}\,\phi_0 - \rho_0) - \frac{1}{2}\mathrm{Tr}\ \ln K \tag{28}$$

which contains Hartree and Fock results.[3)]

In order to investigate Eq. (28) further the two-body potential must be explicitly given and the eigenvalue of the matrix K should be found. These will be published elsewhere.

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Reference

1) M. Stone, "Bosonization", Singapore, World Scientific,1994.
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3) J. W. Negele, "The mean field theory of nuclear structure and dynamics", Rev. Mod. Phys. 54, 1982, pp. 913-1015.
4) H. Kleinert, "Collective quantum fields", Fort. der Phys. 26, 1978, pp. 565-671.
5) See for example Refs. 4) and 3).