Duality Transformation on a Non-Cubic Lattice

Seiju Ami※

1. Introduction

Interacting models on a lattice are at times easier to understand in dual versions.[1] There are two good reasons for this. Firstly, the dual version could show explicitly local excitations responsible for changes of state. Secondly, it could relate states in a weakly interacting region and in a strongly interacting region. Knowledge of a state in either of the two regions can be used to clarify the state in the other region.

In this paper the duality transformation is extented for a lattice of non-cubic symmetry and the systematics of the transformation is explained. This extension is necessary because in the real world non-cubic symmetry is ubiquitous. So far the duality has been exploited in spin models, gauge theories[2] and string theories,[3] and has brought good understanding of these theories. The lattice, however, has been assumed almost always to have cubic symmetry.

2. Dirichlet and Vorono construction

The construction has been used before for analyzing random lattices.[3]

2.1 Geometry

Consider a two-dimensional lattice (Fig. 1) with lattice points $\{M_i\}$ at coordinates $\mathbf{x}_i$. Against this direct lattice L the dual lattice $\tilde{L}$ can be constructed according to Dirichlet and Voronoï (or Wigner and Seitz in physics literature).[3] Place a cell C_i enclosing each lattice point M_i so

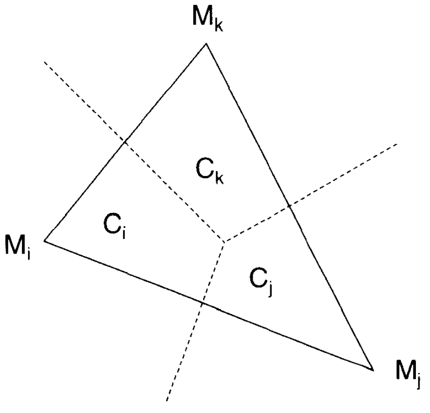

Fig. 1 Direct and dual lattice

※Aomori Public College

that any point within the cell C_i is closer to the point M_i than any other lattice point. As a result any link connecting nearest neighbours such as M_iM_j is cut perpendicularly by one of the edges of the cell C_i. These cells then form the dual lattice.

The direct lattice is characterized by

$$l_{ij} \tag{1}$$

the link distance (1-simplex) connecting M_iM_j and

$$l_{ijk} \tag{2}$$

the area of the 2-simplex formed by $M_iM_jM_k$. It is assumed that a 0-simplex satisfies

$$l_i=1. \tag{3}$$

The dual lattice, on the other hand, is described conveniently by

$$\sigma_i \tag{4}$$

the area (2-cell) of the cell C_i and

$$\sigma_{ij} \tag{5}$$

the length of the edge (1-cell) perpendicular to the link M_iM_j. The point shared by three adjacent cells is called a vertex. The vertex is also a meeting point of three neighbouring edges.

2.2 Dynamical variables

To simplices and cells are assigned the dynamical variables, forms and densities respectively.[3)] The set of 0-forms

$$\left\{\varphi_i\right\} \tag{6}$$

is ascribed to the 0-simplices and is denoted by L_0. Similarly the sets of 1- and 2-forms which are both antisymmetric,

$$\left\{\varphi_{ij}\right\} \tag{7}$$

and

$$\left\{\varphi_{ijk}\right\} \tag{8}$$

are associated with the 1- and 2-simplices and are called L_1 and L_2 respectively. The 2-densities

$$\left\{\psi_i\right\} \tag{9}$$

set of which is $\tilde{L}_2$ are attached to the positively oriented 2-cells. Similarly, the set of 1-densites

$$\left\{\psi_{ij}\right\} \tag{10}$$

which is named $\tilde{L}_1$ is attributed to the oriented 1-cells. On vertices are placed the 0-densities

$$\left\{\psi_{ijk}\right\} \tag{11}$$

set of which is denoted by $\tilde{L}_2$.

It is useful to note that the simplices are averages of fields defined in the continuum. For example φ_i is the representative of a scaler

$$\varphi(x_i) \tag{12}$$

, φ_{ij} the average of a vector

$$\frac{1}{I_{ij}}\int_i^j dx^{\mu}\varphi_{\mu}(x) \tag{13}$$

and φ_{ijk} the average of a tensor

$$\frac{1}{I_{ijk}}\iint_{(ijk)} dx^{\mu}\wedge dx^{\nu}\varphi_{\mu\nu}(x) \tag{14}$$

where *(ijk)* in the last integral signifies a positively oriented triangle.

As a preparation to set up an interaction between neighbouring dynamical variables a scaler product is introduced between forms and densities. They are

$$<\varphi|\psi>=\frac{1}{1!}\sum_i \varphi_i\psi_i \tag{15}$$

$$<\varphi|\psi>=\frac{1}{2!}\sum_{ij}\varphi_{ij}\psi_{ij} \tag{16}$$

$$<\varphi|\psi>=\frac{1}{3!}\sum_{ijk}\varphi_{ijk}\psi_{ijk} \tag{17}$$

where ij and ijk denote nearest neighbour pairs and triangles.

Furthermore a one-to-one correspondence can be defined between L_p and $\tilde{L}_{d-p}$.This is the duality that φ_{ijk} has the dual partner

$$\tilde{\varphi}_{ijk\cdots}=\sigma_{ijk\cdots}I_{ijk\cdots}\varphi_{ijk\cdots}. \tag{18}$$

2.3 Difference operators

Finally operators are introduced on a lattice which are the analogs of gradient and divergence in the continuum. Schematically,[3)]

$$\begin{array}{ccc} L_p & \xleftrightarrow{\text{duality}} & \tilde{L}_{d-p} \\ d\downarrow\uparrow d^* & & d^T\downarrow\uparrow d^{T*} \\ L_{p+1} & \xleftrightarrow[\text{duality}]{} & \tilde{L}_{d-p-1} \end{array} \tag{19}$$

where d, d^T are the gradient operators and its tranpose, and d^*,d^{T*} the divergence operators. The operators in Eq. (19) all satisfy

$$\begin{aligned} &d^2=0\\ &{d^*}^2=0\\ &{d^T}^2=0\\ &{d^{T*}}^2=0. \end{aligned} \tag{20}$$

The gradient operator d acts as

$$(d\varphi)_{ij}=\frac{\varphi_i-\varphi_j}{I_{ij}} \tag{21}$$

$$(d\varphi)_{ijk}=\frac{I_{ij}\varphi_{ij}+I_{jk}\varphi_{jk}+I_{ki}\varphi_{ki}}{I_{ijk}} \tag{22}$$

and for $\varphi\in L_2$

$$d\varphi=0. \tag{23}$$

The transpose operator d^T is defined by

$$<\varphi|d^T\psi>\equiv<d\varphi|\psi>. \tag{24}$$

Thus

$$(d^T\psi)_i=\sum_{j(i)}\frac{\psi_{ij}}{l_{ij}} \tag{25}$$

$$(d^T\psi)_{ij}=\sum_{k(ij)}\frac{l_{ij}}{l_{ijk}}\psi_{ijk} \tag{26}$$

where the summation must be done so that ij and ijk correspond to an edge and a vertex respectively. The divergence operator d^* is defined via the duality as shown in Eq. (19). For example d^* which acts as the transformation from L_1 to L_0 can be constructed by the following steps. First, by duality φ_{ij} is transformed into $\tilde{\varphi}_{ij}$ which second, the operator d^T converts to $(d^T\tilde{\varphi})_i$, an element of $\tilde{L}_2$ and finally this element is changed to an element of L_0. The result is

$$(d^*\varphi)_i=\frac{1}{\sigma_i}\sum_{j(i)}\sigma_{ij}\varphi_{ij}. \tag{27}$$

Similarly the operator d^{T*} can be defined by duality,

$$(d^{T*}\psi)_{ijk}=\sigma_{ijk}\left(\frac{\psi_{ij}}{\sigma_{ij}}+\frac{\psi_{jk}}{\sigma_{jk}}+\frac{\psi_{ki}}{\sigma_{ki}}\right). \tag{28}$$

3. Villain model

Let us apply the method in the previous sections to Villain model[1] which approximates the behaviour of XY model for the large coupling constant.[4] The partition function of the model goes as

$$Z=\left(\prod_{(ij)}\sum_{n_{ij}=-\infty}^{\infty}\right)\left(\prod_i\int_{-\pi}^{\pi}\frac{d\theta_i}{2\pi}\right)e^{-A} \tag{29}$$

where θ_i is the angle variable at the i-th lattice site, n_{ij} the integer field associated to the link (ij), β the coupling strength, and the action is defined by

$$A=\frac{\beta}{2}\sum_{(ij)}(\theta_i-\theta_j-2\pi n_{ij})^2. \tag{30}$$

This partition function can be transformed by Poisson summation formula as

$$Z=\left(\prod_{(ij)}\sum_{b_{ij}=-\infty}^{\infty}\right)\left(\prod_{i}\int_{-\pi}^{\pi}\frac{d\theta_i}{2\pi}\right)\frac{1}{\sqrt{2\pi\beta}}e^{-A'} \tag{31}$$

where the action is given by

$$A'=\sum_{(ij)}\left\{\frac{1}{2\beta}b_{ij}^{2}-ib_{ij}(\theta_i-\theta_j)\right\}. \tag{32}$$

The action in Eq. (32) can be interpreted according Section 2. The second term in the curly bracket contains a derivative of angles,

$$\theta_i-\theta_j=(d\theta)_{ij} \tag{33}$$

according to Eq. (21) (from now on $I_i...=\sigma_j...=1$ for simplicity). Hence the action A' in Eq. (32) takes on the form

$$A'=\frac{1}{2\beta}<\tilde{b}|b>-i<d\theta|b>. \tag{34}$$

In this expression it is obvious that the integer fields b_{ij} are the dual fields and reside on edges of the dual lattice. The derivative in the second term in Eq. (34) can be transposed according to Eq. (24) so that

$$<d\theta|b>=<\theta|d^Tb>. \tag{35}$$

Then the angle integral in Eq. (31) can be performed. The partition function in Eq. (29) is now expressed entirely in terms of the dual fields b_{ij},

$$Z=\left(\frac{1}{\sqrt{2\pi\beta}}\right)^N\left(\prod_{(ij)}\sum_{b_{ij}=-\infty}^{\infty}\right)\delta_{d^Tb,0}e^{-A''} \tag{36}$$

where N is the total number of links and

$$A''=\frac{1}{2\beta}<\tilde{b}|b>. \tag{37}$$

The transformation of the theory from Eq. (29) to Eq. (36) is called the duality transformation.[1),2)] Note that in the dual form the interaction strength appears reversed as in Eq. (36).

The theory in Eq. (36) can be treated entirely in the dual lattice. The constraint in Eq. (36) can be solved easily by

$$b = d^T a \tag{38}$$

because of Eq. (20).

4. Conclusion

The duality transformation for non-cubic symmetry was possible because the action in Eq. (30) was represented by the operator d in Eq. (21),

$$A = \frac{\beta}{2} \sum_{(ij)} \left\{ (d\theta)_{ij} - 2\pi n_{ij} \right\}^2 . \tag{39}$$

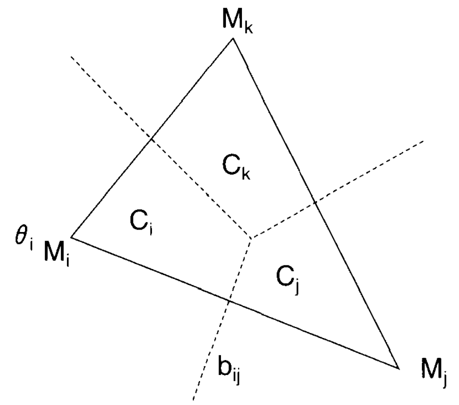

Fig. 2 Dynamical variables

This indicates that once a theory is expressed in terms of the operators in Eq. (19) the duality transformation could be done for a non-cubic lattice. The melting model,[5] for example could be extended in this way for non-cubic symmetry.

Received: May 8, 1996

Reference

1) R. Savit, "Duality in field theory and statistical systems", Rev. Mod. Phys. 52,1980, pp. 453-487.

2) B. Sathiapalan, "Duality in Statitical Mechanics and String Theory", Phys. Rev. Lett. 58, 1987, pp. 1597-1599.

3) C. Itzyckson, "Fields on a random lattice", the talk given at the 1983 Cargèse Summer School on "Progress in Gauge Fields Theory".

4) S. Ami and H. Kleinert, "Vortex contribution to the specific heat in the two-dimensional XY model", Phys. Rev. B 33, 1986, pp. 4692-4705.

5) S. Ami, T. Hofsäss and R. Horsley, "Thermal properties of defect melting", Phys. Lett. 101A, 1984, pp. 145-150.