
Bargaining and Nucleolus in Common Cost Allocation

Masaaki Aoki*

INTRODUCTION

There are many studies in which researchers have tried to apply cooperative game solutions to common cost allocation. Many of these studies focus on Shapley value¹⁾ and only a few researches discuss nucleolus in common cost allocation. The latter studies compare the properties of nucleolus allocation with Shapley value and other allocation schemes, so they do not examine the properties of nucleolus itself in common cost allocation.²⁾

It is well-known that the nucleolus is a subset of the kernel and the kernel is a subset of the bargaining set. This property is a basic one that specifies these three cooperative game solutions. We should investigate the meaning of this property to examine the properties of the nucleolus in common cost allocation. If we can interpret the nucleolus in the bargaining processes for allocating common cost meaningfully, the nucleolus will be a useful allocation scheme in the case where common cost is allocated through the bargaining of the service users.

In Section 1, we describe the model that we will use in this article. This model is proposed in Aoki[1997] and specifies the case where we can formulate common cost allocation as a characteristic function form game. We develop our discussions in this article based on this model.

We define the cooperative game solutions that are used in the later analysis of this paper in Section 2. These solutions are the core, the bargaining set, the kernel, and the nucleolus. We define these cooperative game solutions along the definitions in game theory as much as possible to interpret the meanings of these solutions correctly in common cost allocation.

The game specified in Section 1 is a convex game.³⁾ A convex game has interesting properties and these properties are useful in our analysis. We explain the properties of cooperative game solutions in the convex game in Section 3.

We propose numerical examples that describe the common cost allocation in Section

1) Shapley value is a cooperative game solution proposed in Shapley[1953].

2) Aoki[1996] surveys the past studies that applied cooperative game solutions to common cost allocation and describes the applicability of the nucleolus to common cost allocation.

3) See Aoki[1997]. We define the convex game in Section 1.

4. Using these examples, we will give the nucleolus some meaningful interpretations in the bargaining processes for allocating common cost. We summarize the contents of this article and denote the conclusions of this article in the last section. We will mention the topics that we have to solve in the future study of the nucleolus allocation.

1. Model and Definitions

Aoki[1997] examines the characteristic function in common cost allocation. It specifies the situations where we can estimate characteristic functions from the cost information about the common service. We can apply the characteristic function form game to common cost allocation properly in this model.

The purpose of this article is to examine the nucleolus as a common cost allocation scheme in terms of the bargaining process. So we proceed our analysis based on the model in Aoki[1997]. We explain the essential points of the model in Aoki[1997] here.

Consider the situations where there are some divisions or departments in a firm and they utilize the common services. Common cost is the cost for acquiring the service jointly.⁴⁾ The objective of common cost allocation game is to allocate common cost with a satisfactory manner to divisions or departments.⁵⁾

N is the set of the players. We call N a grand coalition. As the players are rational decision makers, the game players in common cost allocation are the managers of divisions or departments in a firm. It is assumed implicitly that every player prefers to lower allocated cost in the bargaining process for allocating common cost. The subset of N is referred to as a coalition. It is supposed that coalitions can make decisions regarding the acquisition and utilization of the service.⁶⁾ For example, players can decide whether they get the necessary service internally or externally. It is assumed that a coalition makes its decision as if it were one player.⁷⁾

q is the demand of the service, so q_i is the player i 's demand of the service. $C(q)$ is the cost function of the service. When $C(q)$ is a concave function, we can estimate the characteristic function as the maximin value of the benefit arising from the joint acquisition of the service.⁸⁾ The cost function $C(q)$ includes the information about the cost of the external the

4) See Aoki[1997] as to the definition of common cost.

5) This idea leads to "mutually satisfactory allocation" in Thomas[1971].

6) See Aoki[1997](p.75).

7) This assumption is necessary when a characteristic function is estimated. See Aoki[1997](p.4) as to the implications of this assumption.

8) Aoki[1997] examines the relationship between the cost function and the characteristic function. When a cost function is convex, it is necessary to assume another function similar to the characteristic function. Although it is not impossible to estimate a characteristic function, this procedure is complicated and troublesome.

external service when there are some external vendors providing the necessary service in the market.⁹⁾

A characteristic function is a mapping 2^n -dimensional space into real number R . The value of the characteristic function $v(S)$ is the worth that a coalition S can receive.¹⁰⁾ It is rational to assume that the assumption of transferable utilities is satisfied in common cost allocation.¹¹⁾ It is convenient to define the characteristic function as the cost saving game in common cost allocation.¹²⁾ We will proceed our analysis with the following characteristic function.

$$\begin{aligned} v(\emptyset) &= 0 \\ v(S) &= \sum_{i \in S} C(q_i) - C(\sum_{i \in S} q_i) \quad \forall S \subset N \end{aligned} \quad (1)$$

Let $\psi(v) (\subset R^n)$ be an arbitrary cooperative game solution and $\psi_i(v)$ be player i 's value of the game, namely, the allocated amount of benefit to player i . We may represent $\psi_i(v)$ as x_i for abbreviation. As the characteristic function in (1) is 0-normalized, $v(\{i\})=0 (\forall i \in N)$. Let a_i be the allocated cost to player i . We can represent the relationship between a_i and $\psi_i(v)$.

$$a_i = C(q_i) - \psi_i(v) \quad (2)$$

(2) says that the allocated cost to player i (a_i) is automatically determined if we get the value of the game ($\psi_i(v)$). We will often refer to the value of the game in the successive analysis. It should be noted that specifying $\psi_i(v)$ is equivalent to specifying a_i . We will abbreviate summation as follows.

$$q(S) = \sum_{i \in S} q_i \quad \forall S \subset N \quad (3)$$

When the cost function is a concave function, the characteristic function defined in (1) becomes a convex game.¹³⁾ A convex game is a class of game proposed by Shapley[1971]. The definition of this game is :¹⁴⁾

$$v(S) + v(T) \leq v(S \cup T) \quad \forall S \subset N \quad (4)$$

9) The cost function $C(q)$ is the same as the one defined in Aoki[1997](p.18).

10) See Owen[1995](p.213) as to the definition of the characteristic function.

11) See Aoki[1997](p.77) as to the meaning of this assumption.

12) See Aoki[1997](p.90).

13) See Aoki[1997](p.90).

14) Shapley[1971](p.12).

(4) is equivalent to (5).¹⁵⁾ This formula gives us the interpretation for a convex game.

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \quad \forall i \in N \text{ and } \forall S \subset T \subset N - \{i\} \quad (5)$$

It is clear that (5) means the scale of economy. We will examine the case where there is some cost saving from the joint acquisition of the service.

2. Preliminary : Cooperative Game Solutions

We will use various cooperative game solutions in the later sections. It is convenient to define these solutions (the imputation, the core, the bargaining set, the kernel, and the nucleolus) here for our analysis.

2.1. The Imputation and the Core

Let $x = (x_1, x_2, \dots, x_n)$ be a vector denoting the allocated benefit to players. We define the imputation as :

Definition 1: Imputation ¹⁶⁾

$$x_i \geq v(\{i\}) \quad \forall i \in N \quad (6)$$

$$x(N) = v(N) \quad (7)$$

It is clear that (7) represents Pareto Optimality. As (7) is equivalent to $a(N) = C(q(N))$ from (2), (7) means that all cost is allocated to players. (7) is equivalent to $a_i < C(q_i)$. Hence, it means that the allocated cost to each player a_i is less or equal to the individual cost $C(q_i)$ which is the cost when player i gets the service individually.

If an allocation scheme proposes a solution that does not belong to the imputation, it is easy to conceive that player will obtain its service separately. Players will not cooperate to get the service jointly in this case. Therefore, we regard imputation as the minimum requirement for the acceptance of the allocation. We do not mention the imputation directly in the later but the imputation is significant because the core, the bargaining set, the kernel, and the nucleolus are defined as the subset of the imputation.

15) Shapley[1971](p.13).

16) See Owen[1995](p.214).

17) See Owen[1995](pp.218-219).

Core is defined as :

Definition 2: *Core*¹⁷⁾

$$x(S) \geq v(S) \quad \forall S \subset N \quad (8)$$

$$x(N) = v(N) \quad (9)$$

(9) is the same as (7). (8) is equivalent to $a(S) \geq C(q(S))$ by (2). (8) is the extension of (6) in the sense that the conditions of individual players are extended to the conditions of coalitions. (9) means that the allocated cost to coalition S is less or equal to the cost $C(q(S))$, which is the amount when the players in coalition S obtain the service jointly. Therefore, coalition S will not accept the allocation that does not satisfy the core condition. We can regard the core condition as the requirement for the coalition's acceptance of the allocation.

2.2. The Bargaining Set

It is necessary to define the coalition structure before we define the bargaining set. The coalition structure β is a partition of the player set N . Namely,

$$\beta = [B_1, B_2, \dots, B_m]$$

$$\text{where } B_p \cap B_q = \emptyset (p \neq q) \text{ and } B_1 \cup B_2 \cup \dots \cup B_m = N$$

As the cost function is concave in our model, the minimum cost of the service is achieved when the coalition structure $\beta = \{\{N\}\}$ is formed. We focus on the allocation of $C(q(N))$ in our analysis, so we examine the case where the grand coalition is formed.

Some players may propose the coalition structures apart from the grand coalition in the bargaining processes. Players always make such proposal to receive favorable allocation. But the coalition structure other than the grand coalition will not be formed if the negotiation is not broken down. Because the minimum cost of the service is $C(q(N))$.

The bargaining set is defined as the objection and the counter-objection. The objection and the counter-objection are the bargaining rules specifying the bargaining set. The definition of the objection is :

Definition 3: *Objection*¹⁸⁾

Let x be an imputation in a game $(N;v)$ for a coalition structure β . Let k and l be two distinct players in a coalition B of β . An objection of k against l at x is a pair $(C;y)$, satisfying:

18) Maschler[1992](p.596).

- (i) $C \subset N, k \in C, i \notin C$;
- (ii) $y \in R^C, y(C) = v(C)$;
- (iii) $y_i > x_i, \forall i \in C$.

We consider the objection of player k against player l . This objection will succeed if player k can propose the formation of coalition C without player l and can offer the allocation assuring that all the members in coalition C can receive more than x_i . Furthermore, the proposal of player k must be feasible, namely, the new proposal must satisfy $y(C) = v(C)$. The definition of the counter objection is:

Definition 4: *Counter-objection*¹⁹⁾

Let $(C;y)$ be an objection of k against l at $x, x \in X(\beta), k, l \in B \in \beta$. A counter-objection to this objection is a pair $(D;z)$, satisfying:

- (i) $D \subset N, l \in D, k \notin D$;
- (ii) $z \in R^D, z(D) = v(D)$;
- (iii) $z_i \geq y_i, \forall i \in D \cap C$;
- (iv) $z_i \geq x_i, \forall i \in D - C$.

The counter-objection is a rational opinion against the objection. We explain the rationality of the counter-objection as follows.

Player l proposes coalition D without player k . Player l offers the amount more than or equal to the proposal of player k to the members who belong to coalition C and coalition D . Player l also offers the amount more than or equal to the original proposal x to the members who belong to coalition D and do not belong to coalition C . Player l 's proposal must be feasible in order for its proposal to be effective, namely, $z(D) = v(D)$. If player k knows that player l proposed the counter-objection against player k 's objection, player k will not propose the objection against the original allocation x .

We can define the bargaining set using the objection and the counter-objection.

Definition 5 : *Bargaining Set*²⁰⁾

Let $(N;v)$ be a cooperative game with side payments. The bargaining set $M_1(\beta)$ for

19) Maschler[1992](p.596).

20) Maschler[1992](p.559).

a coalition structure β is

$$\begin{aligned} M_1^i(\beta) &:= \{x \in X(\beta) : \text{every objection at } x \text{ can be contered}\} \\ &:= \{x \in X(\beta) : \text{there exists no justified objection at } x\} \end{aligned} \quad (10)$$

The first equation of the above definition says that some counter-objections will be proposed certainly even if a player proposes an objection at the original proposal because this player does not satisfy the original proposal. The second equation of (10) says that no player can propose rational objections at the original proposal.

If an allocation is in the bargaining set, the allocation is stable in the above sense. We can interpret the bargaining set as the bargaining rule of the formulation of the objection and the counter-objection.

2.3. The Kernel

It is necessary to define the excess and the surplus before defining the kernel. The excess is defined as :

Definition 6: *Excess*²¹⁾

Let x be an imputation in a game $(N;v)$ for an arbitrary coalition structure. The excess $e(S,x)$ of a coalition S at x is $v(S) - x(S)$ if $S \neq \emptyset$, and 0 if $S = \emptyset$.

We can regard the excess $e(S,x)$ as the dissatisfaction of coalition S with the allocation x . For example, consider the case where the value of the excess $e(S,x)$ is positive. While coalition S can obtain $v(S)$ by itself, the allocated amount to this coalition is $x(S)$ in this case, which is smaller than $v(S)$. It is natural that the dissatisfaction becomes bigger as $e(S,x)$ increases.

We can give the excess $e(S,x)$ another interpretation. Namely, the excess $e(S,x)$ is the loss of coalition S when coalition S does not accept the allocation x . Consider the case where the value of the excess is negative, and we can understand this interpretation concretely. The definition of the surplus is :

Definition 7: *Surplus*²²⁾

Let x be an imputation in a game $(N;v)$. Let k and l be two distinct players in N . The

21) Maschler[1992](p.596).

22) Maschler[1992](p.603).

surplus of k against l at x is :

$$s_{k,l} := \max_{k \in S, l \notin S} e(S, x) \quad (11)$$

We consider the meaning of the surplus $s_{k,l}(x)$ in terms of the bargaining between player k and player l . Player k refers to the coalitions consisting of player k and the players other than player l in the bargaining processes. We call this collection of coalitions coalitions $D_{k,l}$. Player k looks for the coalition in which the allocation to player k is the most unfavorable in $D_{k,l}$. This coalition gives player k the value of $s_{k,l}(x)$: for example, consider the case where $s_{k,l}(x)$ is positive. As this coalition receives no more than $v(S)$, player k does not satisfy the allocation x and player k thinks that player l receives too much. Player k will insist that player l should transfer some amount to player k . The surplus $s_{k,l}(x)$ gives player k the upper limit of its demand to player l . If player k demands more than $s_{k,l}(x)$, player k cannot show any rational reason to player l because this request means that player k receives more than $v(S)$.

Next, consider the case where $s_{k,l}(x)$ is negative. The excess $e(S,x)$ is the opportunity loss of coalition S if coalition S does not accept allocation x . Therefore, we can regard the surplus $s_{k,l}(x)$ as the minimum opportunity loss of coalition S . Player k will not demand more than $s_{k,l}(x)$ in the bargaining with player l . It should be noted that $s_{k,l}(x)$ is the minimum opportunity loss because the validity of player k 's proposal is lost if its proposal is based on the maximum opportunity loss in the bargaining processes. We can define the kernel using the surplus.

Definition 8: Kernel and Prekernel²³⁾

Let $(N;v)$ be a game and β be a coalition structure. The kernel $K(\beta)$ for (β) is

$$K(\beta) := \{x \in K(\beta) : s_{k,l}(x) > s_{l,k}(x) \Rightarrow x_l = v(\{l\}), \text{ all } k, l \in B \in \beta, k \neq l\} \quad (12)$$

The prekernel $PK(\beta)$ for β is

$$PK(\beta) := \{x \in X^0(\beta) : s_{k,l}(x) = s_{l,k}(x), \text{ all } k, l \in B \in \beta, k \neq l\} \quad (13)$$

Suppose that x is the kernel and consider the case where the surplus of k against l at x is larger than the surplus of l against k at x , namely, $s_{k,l}(x) > s_{l,k}(x)$. Player k may think that player k can demand some amount to player l in this case. But player k cannot make any proposal to player l because player l receives no more than $v(\{l\})$ and player l cannot give

23) Maschler[1992](p.603).

player k any more. It is clear that player k demands nothing to player l if the condition (12) is satisfied. The allocation in the kernel is in equilibrium in this case

The condition (12) is equivalent to :

$$K(\beta) := \{x \in K(\beta) : x_l > v(\{l\}), \text{ some } k, l \in B \in \beta, k \neq l \Rightarrow s_{k,l}(x) \leq s_{l,k}(x)\} \quad (14)$$

Player k cannot demand anything from player l even if player l receives more than $v(\{l\})$ in this case. Because the surplus of k against l at x is not more than the surplus of l against k at x . In other words, the amount of dissatisfaction of player k is smaller than that of player l . The kernel gives players an equilibrium point in the above sense.

It is clear from the definition of the prekernel that the surplus for every pair $(k,l; k \neq l)$ is equal. Therefore the prekernel is in equilibrium. The implication of the prekernel is easy to understand intuitively compared with the kernel. This is the advantage point of the prekernel.

We can get the conclusion that the kernel and the prekernel is the formulation of the equilibrium of the surplus (the measure of dissatisfaction or the opportunity loss) in the bargaining. We will explain the relationship between the kernel and the prekernel in the convex game in the next section.

2.4. The Nucleolus

We have to define the vector $\theta(x) (\in R^{2^n})$ to define the nucleolus.

$$\theta(x) := (e(S_1, x), e(S_2, x), \dots, e(S_{2^n}, x)) \quad (15)$$

The order of elements of $\theta(x)$ is in decreasing order. Namely, if $i < j$, $e(S_i, x) \geq e(S_j, x)$. We explain the operator \rightarrow , which compares any two terms using the lexicographic order. $\theta(x) \rightarrow \theta(y)$ means that there exists some positive integer q such that $\theta_i(x) = \theta_i(y)$ for $i < q$ and $\theta_q(x) < \theta_q(y)$.²⁴⁾ We can define the nucleolus as :

Definition 9: Nucleolus²⁵⁾

Let X be an arbitrary nonempty closed set in R^n . The nucleolus of X - denoted $Nu(X)$, or $Nu(N; v; X)$ - is the set of vectors in X whose θ 's are lexicographically least; i.e.,

$$Nu(x) := \{x \in X : \theta(x) \rightarrow \theta(y), \text{ all } y \in X\}$$

24) We can define \rightarrow as not \leftarrow .

25) Maschler[1992](p.611).

If $X = X(\{N\})$, the nucleolus is called the nucleolus of the game. If $X = X(\beta)$, it is called the nucleolus of the game for the coalition structure β .

The purpose of this article is to examine the property of the nucleolus in common cost allocation. So, we will show some significant implications of the nucleolus in the later section. We describe the intuitive interpretation of the nucleolus here.

We can easily find from the definition of the nucleolus that the nucleolus gives us the vector in which the maximum excess is minimized. Therefore, if we regard the excess as the dissatisfaction measure, we can regard the nucleolus as the formulation of the bargaining process in which maximum dissatisfaction is minimized.²⁶⁾

3. Convex Game and its Implications

We investigate the meanings of the nucleolus based on the results obtained in the field of game theory. We summarize the relationship between the cooperative game solutions defined in the previous section.

It is clear from the definition that the core is a subset of the imputation. Generally speaking, it is not true that the core always exists in any kind of game. But it is a well-known fact that the core exists in a convex game.²⁷⁾ Therefore, we can proceed our discussion supposing the existence of the core. It is clear that there exists an imputation in our model. We examine the subset of the imputation in the later analysis.

It is desirable to propose a core solution if the core exists.²⁸⁾ Consider the allocation that does not belong to the core, and we can understand this reason. Players will form the coalition other than the grand coalition if the allocation is not in the core. As a result, the coalition structure other than $\{\{N\}\}$ is formed. This situation is not desirable because the cost of the service is not minimized.

We explain the relationship between the core and the nucleolus here. Generally speaking, the nucleolus is included in the core if the core exists.²⁹⁾ The nucleolus is one point solution.³⁰⁾ Considering these points, the nucleolus is useful to select one point from the

26) See Maschler[1992](p.611) as to the intuitive interpretation of the nucleolus and the difficulties of the lexicographic order comparison. Hamlen et al.[1977](p.622) points out that the nucleolus allocation is similar to "Justice" in Rawls[1971].

27) See Shapley[1971](p.21).

28) Hamlen et al.[1977] evaluates four allocation schemes (Shapley value, nucleolus, activity level, and Moriarity) in terms of the core.

29) See Maschler[1979](p.335). Strictly speaking, the nucleolus is included in ϵ -core, which is the extension of the core. As the core always exists in a convex game, the nucleolus is included in the core in our analysis.

30) See Schmeidler[1969](pp.1164).

core.

We notice the well-known results about the bargaining set, the kernel, and the nucleolus.

$$M_1^i(\beta) \supset K(\beta) \supset Nu(X) \quad (16)$$

We can interpret the above relationship as the bargaining process in common cost allocation. We describe the bargaining processes based on (16) generally as follows.

- Step 1:** Players propose some allocation based on the rule of the objection and the counter-objection. Players obtain alternative solutions in the bargaining set.
- Step 2:** Players select some allocations out of the solutions in Step 1 (the bargaining set) using the bargaining rule of the equilibrium of dissatisfaction. Players get the allocation in the kernel or the prekernel.
- Step 3:** Players obtain the unique allocation by the bargaining rule in which the maximum dissatisfaction is minimized. This final result is in the nucleolus.

The bargaining set coincides with the core in a convex game,³¹⁾ therefore, the relationship (16) means (17) in our model.

$$Core(v) \supset K(\beta) \supset Nu(X)^{32)} \quad (17)$$

The relationship (17) tells us that we can specify one point from the range of the core through the bargaining processes described above.

The kernel coincides with the prekernel in a convex game.³³⁾ This fact suggests that (13) is not only the definition of the prekernel but also the definition of the kernel in our model. This gives us an easier interpretation of the kernel in the bargaining process.

The prekernel coincides with the nucleolus in a convex game.³⁴⁾ In our model, we can interpret nucleolus in terms of the bargaining rule in which the dissatisfaction with the allocation is in equilibrium. Such interpretation gives us a more intuitive interpretation than the lexicographic order.

31) Maschler et al.[1971](p.92).

32) $Core(v)$ is the core of the game v .

33) Maschler et al.[1971](p.83).

34) Maschler et al.[1971](p.91).

4. Bargaining Process in Common Cost Allocation

4.1. Numerical Example

We will examine the meanings of the nucleolus in common cost allocation using the departments using the common service.³⁵⁾ We consider the case $N=\{1,2,3\}$ here.

We consider the following piece-wise linear cost function.³⁶⁾

$$C(q) = \begin{cases} 40q & 0 \leq q < 15 \\ 20q + 300 & 15 \leq q < 30 \\ 10q + 600 & 30 \leq q \end{cases}$$

q is the demand of the service. It is clear that the marginal cost of the service decreases as the demand of the service increases. So $C(q)$ is a concave function. We depict $C(q)$ as Figure 1.

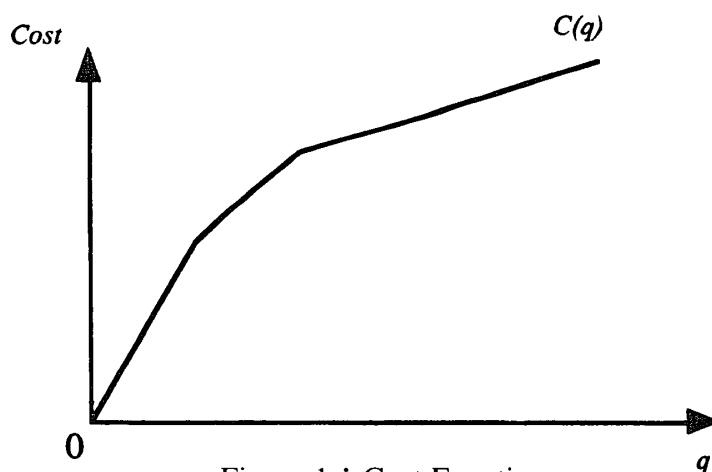


Figure 1 : Cost Function

q_i is player i 's demand of the service. Suppose that $q_1=10$, $q_2=15$, and $q_3=25$. We can estimate the cost of the service for coalition $S(\subset N)$.

$$\begin{aligned} C(\{1\}) &= 400, & C(\{2\}) &= 600, & C(\{3\}) &= 800 \\ C(\{1,2\}) &= 800, & C(\{1,3\}) &= 950, & C(\{2,3\}) &= 1,000 \\ C(\{1,2,3\}) &= 1,100 \end{aligned}$$

We can calculate the characteristic function of our game using (1).

35) It is well-known that the kernel coincides with the nucleolus in a three-person game. We should use the example where players are more than three to make our discussion general. But the kernel and the nucleolus are the same in a convex game. Therefore, it is sufficient to refer to a three-person game example in our model. The results obtained here are applied into n -person case.

36) The results of our analysis are applied to the case where the cost function of the service is concave.

$$\begin{aligned}
v(\{1\}) &= 0, & v(\{2\}) &= 0, & v(\{3\}) &= 0 \\
v(\{1,2\}) &= 200, & v(\{1,3\}) &= 250, & v(\{2,3\}) &= 400 \\
v(\{1,2,3\}) &= 700
\end{aligned}$$

We will examine the implications of the nucleolus in common cost allocation using the above characteristic function.

4.2. Implications of Imputation and Core in Common Cost Allocation

As the game defined above is 0-normalized, the set of the imputation is described as:

$$\begin{aligned}
x_i &\geq 0 & \forall i \in N \\
x_1 + x_2 + x_3 &= 700
\end{aligned}$$

The set of the imputation is the triangle area in 3-dimensional space. Because the hyperplane $x_1 + x_2 + x_3 = 700$ is bounded by $x_i \geq 0 (i=1,2,3)$. We can depict the imputation set as Figure 2.

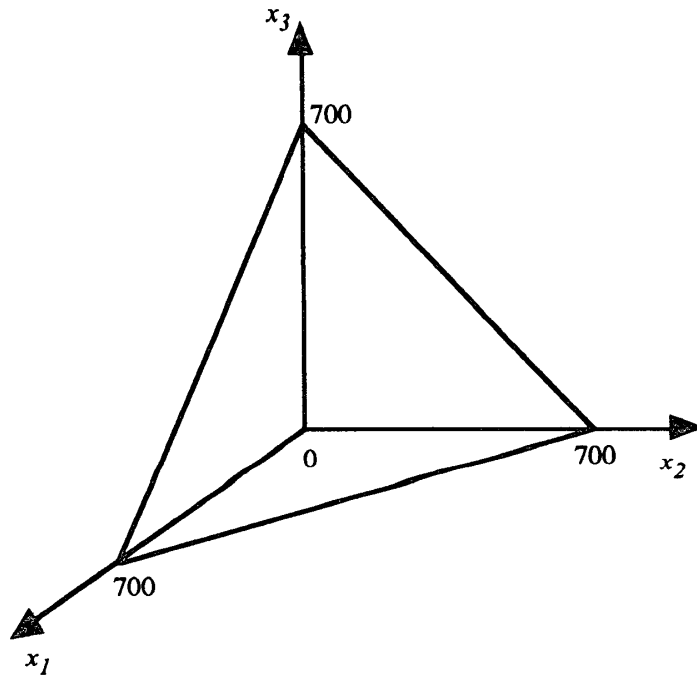


Figure 2 : Imputation

The core of our game is :

$$\begin{aligned} x_i &\geq 0 && \forall i \in N \\ x_1 + x_2 &\geq 200 \\ x_1 + x_3 &\geq 250 \\ x_2 + x_3 &\geq 400 \\ x_1 + x_2 + x_3 &= 700 \end{aligned}$$

We can depict the core as a polygon(ABCDEF) in Figure 3.

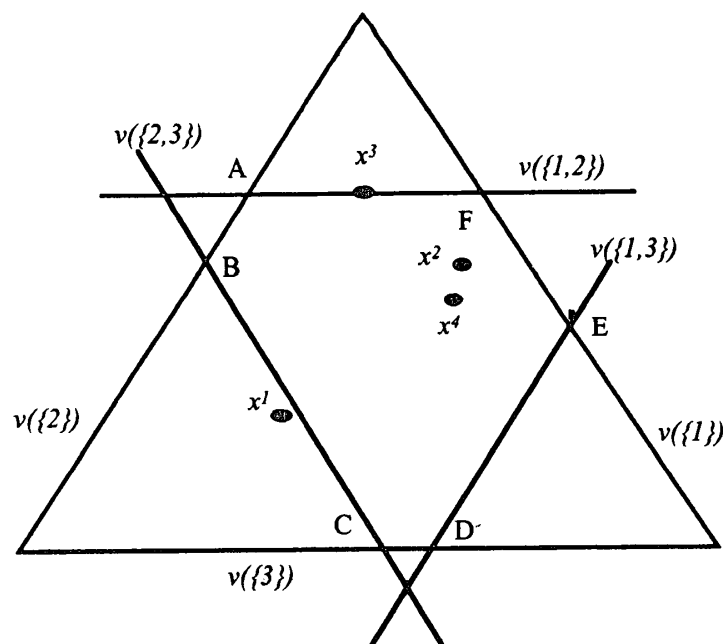


Figure 3 : Core

The triangle area in Figure 3 corresponds to the triangle area in Figure 2. So we can represent the core in two-dimensional space in a three-person game.

4.3. The Bargaining Set and the Core in Common Cost Allocation

We examine the meanings of the bargaining set in terms of the core. For example, the allocation $x^1 = (310, 190, 200)$, which is depicted in Figure 3, is proposed in the bargaining for allocating common cost. It is clear that x^1 is not in the core. Hence, x^1 does not belong to the bargaining set.

Player 2 and player 3 are not satisfied with x^1 because they receive less than $v(\{1,2\})$. They may form coalition $\{1,2\}$. Player 2 will propose the objection against player 1. Namely, player 2 forms coalition $\{2,3\}$ and offers $y_2 = 195$ and $y_3 = 205$.³⁷⁾

37) Player 3 can also propose the same objection against player 1. There is no need that $y_2 = 195$ and $y_3 = 205$. Any objection satisfying $y_2 > 190$, $y_3 > 200$, and $y_2 + y_3 = 400$ is relevant to this case.

It is easy to check that the proposal of player 2 satisfies the conditions of the objection in Definition 3.

When player 2 proposes this objection against player 1, player 1 cannot propose any rational counter-objection against player 2. Player 1 cannot offer a proposal z in which $z_1 > 210$ and $z_3 > 205$ because $v(\{1,3\}) = 250$. This example says that the players who receive an unfavorable allocation can propose the rational objection in the bargaining processes.

The coalition structure $\beta^1 = \{\{1\},\{2,3\}\}$ may be formed in the above example. Because players cannot propose any counter-objection, the objection proposed by player 2 is admitted. The resulting allocation is not desirable as a firm. For example, consider the case where there are some external vendors providing the service. The coalition structure $\beta^1 = \{\{1\},\{2,3\}\}$ means that player 1 gets the service externally and coalition $\{2,3\}$ obtains the internal service.³⁸⁾ The resulting cost allocation is $a = (400,405,595)$. It is clear that $a(N)$ is more than $C(N)$.³⁹⁾ Namely, the minimum cost of the service is not achieved in this case.

Next, suppose that the allocation $x^2 = (100,250,350)$ is proposed in the bargaining process (See Figure 3). It is clear that x^2 is in the core of our game. Hence, x^2 belongs to the bargaining set. x^2 is a stable allocation in the sense that no player can propose any rational objections.

The analysis here says that the allocation in the bargaining set is useful as an initial proposal in the bargaining process. Because some suboptimal coalition may be formed if the allocation that is not in the bargaining set is proposed.

The property of our model, *i.e.*, the bargaining set coincides with the core, is advantageous when we select an initial proposal. It is difficult to specify the range of the bargaining set in general but we can easily check whether an allocation is in the core or not. In our model, selecting the core allocation is equivalent to selecting the allocation in the bargaining set. We can specify the allocation in the bargaining set without complexity in our model.

The analysis here also suggests that another bargaining rule is necessary to allocate common cost in a more satisfactory manner. For example, consider that the allocation $x^3 = (100,100,500)$ is proposed (See Figure 3). It is clear that x^3 is the core allocation. As x^3 is in the bargaining set, no player can object to this allocation. But player 1 and player 2 receives no more than $v(\{1,2\}) = 200$ in this allocation. It does not seem that player 1 and player 2 are satisfied with x^3 . If we confine the bargaining rule to the objection and the counter-objection, we cannot deny the possibility in which x^3 is accepted in the bargaining processes.

38) There is no problem to suppose that player 1 obtains the internal service and coalition $\{2,3\}$ gets the external service in this context.

39) $a(N)=1,400$ and $C(N)=1,100$.

4.4. The Kernel and the Nucleolus in Common Cost Allocation

Suppose that the allocation $x^2 = (100, 250, 350)$ is proposed to examine the meanings of the kernel. It is clear that x^2 is in the bargaining set. We calculate the excess of a coalition S at x^2 using Definition 4.

$$\begin{aligned} e(\{1\}, x^2) &= -100 & e(\{2\}, x^2) &= -250 & e(\{3\}, x^2) &= -350 \\ e(\{1,2\}, x^2) &= -150 & e(\{1,3\}, x^2) &= -200 & e(\{2,3\}, x^2) &= -200 \\ e(\{1,2,3\}, x^2) &= 0 \end{aligned}$$

Consider the bargaining between player 1 and player 3. They calculate $s_{1,3}(x^2)$ and $s_{3,1}(x^2)$, which are defined in Definition 7.

$$\begin{aligned} s_{1,3}(x^2) &= \max \{e(\{1\}, x^2), e(\{1,2\}, x^2)\} = -100 \\ s_{3,1}(x^2) &= \max \{e(\{3\}, x^2), e(\{2,3\}, x^2)\} = -200 \end{aligned}$$

$s_{1,3}(x^2)$ is the minimum loss of player 1 when player 1 does not accept the allocation x^2 and forms a coalition without player 3. We can interpret $s_{3,1}(x^2)$ in the same manner as $s_{1,3}(x^2)$. The opportunity loss of player 1 is smaller than that of player 3 when they do not accept the allocation x^2 .

This result shows that player 1 keeps a more advantageous position than player 3 in the bargaining process. Therefore, player 1 may demand to transfer some amount to player 3. But it is not rational to demand the difference between $s_{1,3}(x^2)$ and $s_{3,1}(x^2)$, *i.e.*, $s_{1,3}(x^2) - s_{3,1}(x^2)$ because, if player 1 demands this amount to player 3, $s_{1,3}(x^2) = -200$ and $s_{3,1}(x^2) = -100$. It is clear that player 3 demands a transfer of some amount to player 1 conversely.

It is rational that player 1 demands the half of $s_{1,3}(x^2) - s_{3,1}(x^2)$ to player 3 if player 1 demands this amount, $s_{1,3}(x^2) = s_{3,1}(x^2)$. Hence, player 3 does not demand anything to player 1.

Player 1's demand to player 3 is 50 in this example. If player 3 accepts this proposal, the new allocation $x^4 = (150, 250, 300)$ is proposed (See Figure 3). We calculate the excess $e(S, x^4)$ of this new proposal.

$$\begin{aligned} e(\{1\}, x^4) &= -150 & e(\{2\}, x^4) &= -250 & e(\{3\}, x^4) &= -300 \\ e(\{1,2\}, x^4) &= -200 & e(\{1,3\}, x^4) &= -200 & e(\{2,3\}, x^4) &= -150 \\ e(\{1,2,3\}, x^4) &= 0 \end{aligned}$$

We obtain $s_{1,3}(x^4)$ and $s_{3,1}(x^4)$ as follows.

$$\begin{aligned} s_{1,3}(x^4) &= \max \{e(\{1\}, x^4), e(\{1,2\}, x^4)\} = -150 \\ s_{3,1}(x^4) &= \max \{e(\{3\}, x^4), e(\{2,3\}, x^4)\} = -150 \end{aligned}$$

As $s_{1,3}(x^4) = s_{3,1}(x^4)$, the surplus of player 1 against player 3 at x^4 is equal to the surplus of player 3 against player 1 at x^4 . There is no room for bargaining between player 1 and player 3.

Next, we consider the bargaining between player 1 and player 2. As $s_{1,2}(x^4) = s_{2,1}(x^4) = -150$, the surplus between player 1 and player 2 is the same. Player 1 demands nothing of player 2, and vice versa. As $s_{2,3}(x^4) = s_{3,2}(x^4) = -200$, the surplus between player 2 and player 3 is also identical.

As $s_{k,l}(x^4) = s_{l,k}(x^4)$ holds for all the possible pair $(k,l;k \neq l)$, $x^4 = (150,250,300)$ is the prekernel of our game. This game is a convex game, so x^4 is the kernel. The numerical example shows that the kernel (prekernel) is useful to select single allocation from the bargaining set in our model. The bargaining process is specified by the equilibrium of the dissatisfaction with the allocation.

The kernel is not a point in general. We can see this from the definition of the kernel but the kernel is the same as the nucleolus in our allocation model, which is a convex game. Therefore, x^4 is the nucleolus of our game. We compare x^2 with x^4 to consider the meanings of the nucleolus. x^2 and x^4 is in the bargaining set but x^2 is not the kernel and x^4 is the kernel.

We compute $\theta(x^2)$ and $\theta(x^4)$ using (15).

$$\theta(x^2) = (0, -100, -150, -200, -200, -250, -350)$$

$$\theta(x^4) = (0, -150, -150, -200, -200, -250, -300)$$

We compare $\theta(x^2)$ and $\theta(x^4)$ in lexicographic order. As $\theta^1(x^2) = \theta^1(x^4)$ and $\theta^2(x^2) > \theta^2(x^4)$, $\theta(x^4) \preceq \theta(x^2)$. The relationship $\theta(y) \preceq \theta(x^4)$ always holds for any imputation y because x^4 is the nucleolus. We examine this point in the bargaining process.

The excess $e(S,x)$ is the dissatisfaction with the allocation x . So the dissatisfaction with x is ordered in decreasing order in $\theta(x)$. The maximum dissatisfaction is minimized as much as possible in the bargaining process.

For example, the coalition that has the biggest dissatisfaction with x^2 is $\{1\}$.⁴⁰⁾ The initial purpose of the bargaining is to minimize player 1's dissatisfaction with x^2 . The bargaining continues until the lexicographic minimum vector is obtained. The results is the nucleolus.

The nucleolus and the kernel are same in our model. It means that we can proceed the bargaining for allocating common cost with the bargaining rule of the kernel (prekernel). In

40) The first element of $\theta(x^2)$ is $e(N,x^2)=0$. As x^2 and x^4 are in the core, all the elements of $\theta(x)$ is not positive. We can neglect $e(N,x^2)$ and $e(N,x^4)$ in the lexicographic order comparison.

our model, we can give the nucleolus two meanings in the bargaining processes. Namely, they are the equilibrium of the dissatisfaction and minimizing the maximal dissatisfaction.

Conclusions

We treat the case where there are services that are used in common by divisions or departments and the cost function of the service is concave. The scale of economy works in this case and users can save cost by the joint utilization of the service.

We notice the fact that the nucleolus is the subset of the kernel and the kernel is the subset of the bargaining set in our analysis. We investigate the meaning of the nucleolus in the bargaining processes based on this fact.

We proceed our analysis based on the model proposed in Aoki[1997]. It says that the common cost allocation setting is formulated as a convex game if the cost function of the service is concave. Furthermore, we notice the following properties of a convex game.

- The core coincides with the bargaining set in a convex game.
- The kernel coincides with the nucleolus in a convex game.
- The prekernel is the same as the kernel in a convex game.

The numerical example in Section 4 shows that the bargaining set is desirable as the initial proposal in the bargaining processes. Generally, the core is the subset of the bargaining set and it is difficult to specify its range but we can obtain the range of the core easily. The analysis of this article says that we can use the core allocation as an initial proposal in the bargaining processes.

If the players of the game use the equilibrium of the dissatisfaction as the rule of the bargaining, the resulting solution is the kernel. It also means that this result is the nucleolus in our model. As the nucleolus is the realization of minimizing the maximum dissatisfaction, players may accept nucleolus as an allocation scheme.

If the players of the game agree with the bargaining rule mentioned in this article, which are the objection and the counter-objection, the equilibrium of the dissatisfaction, and the minimization of the maximum dissatisfaction, players do not need to actually bargain for their allocation. We can use the nucleolus as the final allocation. Therefore, the nucleolus is a promising proxy solution to the bargaining processes.

We can easily obtain the nucleolus by solving a series of linear programming problems.⁴¹⁾ So, the nucleolus is desirable in terms of the applicability to practice.

41) See Kohlberg[1972].

We have described the desirable properties of the nucleolus in the above discussion. But, we have to solve two problems to apply the nucleolus to common cost allocation.

First, we should examine the excess in detail. We have shown that the excess is the measure of the dissatisfaction but there may be other measures of dissatisfaction from the stand point of the users of the service. There are many studies about this topic. We can apply the results of these investigations to our model.⁴²⁾

Second, we do not make a normative approach to the nucleolus in common cost allocation. We describe the properties of the nucleolus in common cost allocation. Namely, our approach is descriptive. We refer to Shapley value, which is a familiar cooperative game solution, to explain this clearly.

It is well-known that Shapley value is derived from a set of axioms and these axioms are the necessary and sufficient conditions for Shapley value. Hence, the acceptance of Shapley axioms is equivalent to the acceptance of Shapley value. Many researchers have interpreted Shapley value axioms in common cost allocation in the past.⁴³⁾

Recently, some game theorists propose a set of axioms specifying the nucleolus.⁴⁴⁾ If we interpret these axioms in common cost allocation properly, we can give the nucleolus as an allocation scheme significant meaning.

As the discussion of these problems is not the purpose of this article, we have not examined these problems here. We will explore these two problems in another paper.

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42) Aoki[1988] surveys the various nucleolus and examines the implications of these in common cost allocation.

43) See Aoki[1996a] as to the past studies in which Shapley value is applied to common cost allocation.

44) See Potters[1991], Snijders[1995], and Sobolev[1995].

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