Various Nucleoli in Common Cost Allocation

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INTRODUCTION

The nucleolus is proposed by Schmeidler[1969] as a cooperative game solution. The nucleolus has desirable properties as a common cost allocation scheme but researchers in the past have not studied the properties of the nucleolus in common cost allocation sufficiently.[1)]

We can interpret the nucleolus in the bargaining processes. The nucleolus is the allocation scheme in which players of a game admit "minimizing the maximum dissatisfaction" as a bargaining rule.[2)] The dissatisfaction measure of the nucleolus is represented by the excess, which is the difference between the worth of the coalition and the allocated amount to the coalition. It is natural that we can conceive the dissatisfaction measure other than the excess in common cost allocation.

Gately[1974] proposes "propensity to disrupt" as a new dissatisfaction measure in a cooperative game. Littlechild and Vaidya[1976] and Charnes and Seiford[1978] extend "propensity to disrupt" and propose the new nucleolus based on Gately's proposal. Young et al. [1980] proposes new nucleolus based on the least core. There are a few studies in which these new solutions are applied to common cost allocation.[3)] The purpose of this study is to examine the properties of the new nucleolus described above.

We formulate common cost allocation setting as a characteristic function form game. So we proceed our discussion using the model in Aoki[1997].

In section 1, we describe the notations and the model. We also explain the computational procedure of the nucleolus in Section 1. Because this procedure applies to other nucleolus that we examine in the later section.

In Section 2, we focus on the nucleolus based on the propensity to disrupt. We describe the nucleolus based on the least core in Section 3.

In section 4, we analyze the properties of the various nucleolus described in the previous

1) Aoki[1997] surveys the literature in which Shapley value and the nucleolus are applied to common cost allocation.

2) See Aoki[1996a].

3) Aoki[1988] examines the properties of the nucleolus based on the propensity to disrupt and the least core.

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sections. We will use the numerical examples in our analysis to make our discussion concrete.

In the last section, we summarize the results of our analysis. We refer to the recent studies that treat with the nucleolus in common cost allocation and show the direction of the future study.

1. Preliminary and the Computation of the Nucleolus

1.1. *Preliminary : Definition and Model*

The purpose of this paper is to apply the cooperative game solutions, which are similar to the nucleolus, to common cost allocation and explore the properties of these solutions. We formulate common cost allocation setting as a characteristic function form game. Aoki[1997] specifies the cases where we can estimate the characteristic function properly in common cost allocation. So, we proceed our analysis with the model described in Aoki[1997]. We give an outline of this model here.[4]

N is the set of the players. We call N a grand coalition. As the players are rational decision makers, the game players in common cost allocation are the managers of divisions or departments in a firm. It is assumed implicitly that every player prefers to lower allocated cost. The subset of N is referred to as a coalition. It is supposed that coalitions can make decisions regarding the acquisition and utilization of the service. For example, players can decide whether they get the necessary service internally or externally. It is assumed that a coalition makes its decision as if it were one player.

q is the demand of the service, so q_i is the player i's demand of the service. $C(q)$ is the cost function of the service. When $C(q)$ is a concave function, we can estimate the characteristic function as the maximin value of the benefit arising from the joint acquisition of the service. The cost function $C(q)$ includes the information about the cost of the external service when there are some external vendors providing the necessary service in the market.

A characteristic function is a mapping 2^n-dimensional space into the real number R. The value of the characteristic function $v(S)$ is the worth that a coalition S can receive.[5] It is rational to assume that the assumption of transferable utilities is satisfied in common cost allocation. It is convenient to define the characteristic function as the cost saving game in common cost allocation. We will proceed our analysis with the following characteristic function.

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4) See Aoki[1997] as to the full detail of the model used in this article.

5) See Owen[1995](p.213) as to the definition of the characteristic function.

$$v(\emptyset) = 0$$
$$v(S) = \sum_{i \in S} C(q_i) - C(\sum_{i \in S} q_i) \qquad \forall S \subset N \tag{1}$$

Let $\psi(v)(\in R^n)$ be an arbitrary cooperative game solution and $\psi_i(v)$ be player i's value of the game, namely, the allocated amount of benefit to player i. We may represent $\psi_i(v)$ as x_i for abbreviation. As the characteristic function in (1) is 0-normalized, $v(\{i\}) = 0(\forall_i \in N)$. Let a_i be the allocatedcost to player i. We can represent the relationship between a_i and $\psi_i(v)$.

$$a_i = C(q_i) - \psi_i(v) \tag{2}$$

(2) says that the allocated cost to player i (a_i) is automatically determined if we get the value of the game ($\psi_i(v)$). We will often refer to the value of the game in the successive analysis. It should be noted that specifying $\psi_i(v)$ is equivalent to specifying a_i.

When the cost function is a concave function, the characteristic function defined in (1) become a convex game.6) A convex game is a class of the game proposed by Shapley [1971]. The definition of this game is :7)

$$v(S) + v(T) \leq (S \cap T) + v(S \cup T) \qquad \forall S \subset N \tag{3}$$

(3) is equivalent to (4).8) This formula gives us interpretation for a convex game.

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \qquad \forall i \in N \; and \; \forall S \subset T \subset N - \{i\} \tag{4}$$

It is clear that (4) means the scale of economy. We will examine the case where there comes some cost saving from the joint acquisition of the service.

We will use the following notation instead of the summation for abbreviation. x is a n-dimensional vector.

$$x(S) = \sum_{i \in S} x_i \qquad \forall S \subset N \tag{5}$$

We define the excess using (5).

$$e(S,x) = v(S) - x(S) \tag{6}$$

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6) See Aoki[1997](p.19).
7) Shapley[1971](p.12).
8) Shapley[1971](p.13).

We can regard the excess as the dissatisfaction of coalition S with the allocation x.[9] Aoki[1996a] interpret the meaning of the nucleolus in the bargaining process for allocating common cost. It is assumed that players bargain each other about the allocation of the common cost in Aoki[1996a]. Though we will not mention the bargaining process explicitly in the later sections, we proceed our discussion under this assumption.

1.2. *The Computation Procedure of the Nucleolus*

We can compute the nucleolus by solving a series of the linear programming problems.[10] The cooperative game solutions, which are similar to the nucleolus, are obtained by solving the linear and non-linear programming problems. We describe the computational procedures of the nucleolus here.

The first stage problem for obtaining the nucleolus is the following.

Problem 1

$$\begin{aligned} &min && r \\ &s.t. && e(S,x) \leq r \qquad \forall S \subset N \\ & && x(N) = (N) \\ & && x \geq 0 \end{aligned}$$

The first constraints of the above problem means that r is the maximum excess. Therefore, the maximum excess is minimized in Problem 1.

If some allocation x is determined uniquely in the first stage problem, this result is the nucleolus. But if not, we have to solve the following second problem.

Problem 2

$$\begin{aligned} &min && r \\ &s.t. && e(S,x) \leq r \qquad \forall S \subset N - B_1 \\ & && x = X^1 \\ & && x \geq 0 \end{aligned}$$

X^1 is the set of the solutions in Problem 1. Let r_1 be the optimal value of r in Problem 1. B_1 is the collection of coalition S such that $e(S,x)=r_1$ *for* $x \in X^1$. If we obtain a unique solution in Problem 2, this is the nucleolus. If not, it is necessary to solve the third stage

9) See Aoki[1996a](pp.6-7) as to the definition and the meaning of the excess.

10) See Kohlberg[1972].

problem. This process continues until we get the unique solution. As the nucleolus is a single point solution and exists in any case, we can obtain the nucleolus certainly after finite repetition.[11)]

The nucleolus based on the propensity to disrupt and the nucleolus based on the least core are obtained by solving a series of the linear and non-linear programming problems. These problems are different from the linear programming problem for the nucleolus in their constraints. It is necessary to solve these problem repeatedly to get these nucleolus. Though we will show only the first stage problem in the later sections, it should be noted that repetitive computational procedures are necessary to get these solutions.

We regard the excess $e(S,x)$ as the measure of the dissatisfaction in the nucleolus. The nucleolus is the solution in which the maximum dissatisfaction is minimized as much as possible. But it is not always true that the coalition S has the excess $e(S,x)$ as the dissatisfaction with the allocation x. Rather, it is natural to think that there are many kind of the dissatisfaction measure in common cost allocation. Gately[1974] call the excess $e(S,x)$ as the dissatisfaction measure into question. The successive studies are inspired by Gately[1974].

2. The Nucleolus Based on Propensity to Disrupt

2.1. *The Equal Propensity to Disrupt Solution*

Gately[1974] proposes "propensity to disrupt" has the dissatisfaction measure with the allocation. Gately[1974] considers the case where player i does not accept some allocation.[12)] Gately[1974] compares the two kind of opportunity losses in this case. One is the loss that is incurred to player i and the other is the loss that is incurred to coalition $N-\{i\}$.

If player i is not satisfactory to some allocation x, player i will not accept this allocation and depart from the grand coalition N. Player i has to obtain the necessary service individually in this case. When player i does not accept the allocation x, the opportunity loss of player i is :

$$x_i - v(\{i\}) \tag{7}$$

If player i departs from the grand coalition, coalition $N-\{i\}$ has to get the necessary service by itself. The opportunity loss of coalition $N-\{i\}$ in this case is :

11) The number of the iteration is at most $n-1$ in n-person game.

12) Thereafter, the allocation means the imputation, namely, $x_i \geq v(\{i\})$ and $x(N)=v(N)$ for all x.

$$x(N-\{i\})-v(N-\{i\}) \tag{8}$$

If the amount (8) is larger than the amount (7), player i thinks that coalition $N-\{i\}$ receives too much. Hence, player i demands that coalition $N-\{i\}$ should transform some amount to player i. Gately[1974] considers that the ratio of (7) to (8) is desirable dissatisfaction measure. This is the propensity to disrupt of player i. Let $d_i(x)$ be the propensity to disrupt of player i with the allocation x.

$$d_i(x)=\frac{x(N-\{i\})-v(N-\{i\})}{x_i-v(\{i\})} \tag{9}$$

It is clear that $d_i(x)$ becomes larger when (8) is relatively larger than (7).[13] Therefore, players will willingly accept the allocation x in which the value of (9) is small. Gately[1974] proposes the solution in which every player's propensity to disrupt is identical to determine a unique solution. We refer to this solution as "equal propensity to disrupt solution."

We can calculate the equal propensity to disrupt solution of n-person game by solving the following simultaneous equations.

Problem 3

$$d_1(x)=d_j(x) \qquad (j=2,\cdots,n)$$
$$x(N)=v(N)$$

As there are n equations for n unknown variable x_i in Problem 3, we can get a unique solution except the special case where there exist some dependent equations in Problem 3.

The bargaining rule of the nucleolus, "minimizing the maximum dissatisfaction," is not fully incorporated into the equal propensity to disrupt solution because the propensity to disrupt (9) is defined on each player.[14] This solution is similar to the prekernel rather than the nucleolus.[15]

We can calculate the equal propensity to disrupt solution more easily than the nucleolus since the former is the solution of the simultaneous equations while the latter is the solution

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13) It is assumed that the value of (7) and (8) are positive in (9). If the core of the game not empty and the core does not consist of a single point, there exists some x such that (9) is positive. This assumption is valid in our model.

14) As the propensity to disrupt is equal for all players, "minimizing the maximum dissatisfaction" is realized in each player. But the propensity to disrupt of coalition is not considered in the equal propensity to disrupt solution.

15) See Aoki[1996a](p.7) as to the definition of the prekernel.

of the linear programming problem. This is an advantage of the equal propensity to disrupt solution over the nucleolus.

If players of the game regard the propensity to disrupt as the dissatisfaction measure and accept the equal propensity to disrupt as the bargaining rule in the bargaining processes for allocating common cost, this solution may give us a promising allocation. Although the solution proposed by Gately[1974] is different from the nucleolus in detail, we have examined Gately's solution here because this solution has desired properties described above and give the foundation to the later studies.

2.2. *The Disruption Nucleolus (LIttlechild and Vaidya)*

Littlechild and Vaidya[1976] points out the following difficulties of the propensity to disrupt proposed by Gately[1974].[16)]

1. It does not specify which size of coalition is the "relevant" one.
2. The solution corresponding to coalition of size k is independent of characteristic function values for all coalitions except those size of k, $n-k$ and n.
3. Although the equal propensity to disrupt solution for k-person coalitions minimizes the maximum propensity to disrupt over all coalitions of size k, there may be coalitions of some other size with a higher propensity to disrupt.
4. An equal propensity to disrupt solution may not be in the core of the game.

The first three difficulties arise from the definition of the propensity to disrupt. That is to say, Gately's propensity to disrupt is defined on each player not on the coalition. Littlechild and Vaidya[1976] generalizes the propensity to disrupt to overcome the above three difficulties. They propose the new propensity to disrupt that is defined on the coalition.[17)]

$$d_i(S,x) = \frac{x(N-S) - v(N-S)}{x(S) - v(S)} \qquad \forall S \subset N (S \neq N, \emptyset) \tag{10}$$

The denominator of (10) denotes the opportunity loss of coalition S when coalition S does not accept the allocation x. The numerator of (10) is the opportunity loss of coalition $N-S$ when coalition S departs from the grand coalition N. It is clear that (10) becomes larger as the dissatisfaction of coalition S grows.

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16) See Littlechild and Vaidya[1976](p.153).

17) The fourth difficulty cannot be resolved even if we adopt (10).

We can define the disruption nucleolus using (10). It is necessary to denote the preliminary notations to define the disruption nucleolus.

$q(x)(\in R^{2^n-2})$ is a vector whose elements are $d^1(S,x)$ and the elements of this vector are in decreasing order. The operator $\prec$ compares any two term using lexicographic order. Namely, if $q(x) \prec q(y)$, there exists some positive integer p such that $q_i(x)=q_i(y)$ for $i<p$ and $q_p(x)<q_p(y)$. We can define $\preceq$ as not $\succ$. We define the disruption nucleolus dn as follows.

$$dn = \{x \in X : q(x) \preceq q(y), \ \ for \ \ all \ \ y \in X\} \tag{11}$$

We can compute the disruption nucleolus by solving a series of the minimization problems. As the propensity to disrupt (10) is not linear, we have to solve the following non-linear programming problem to obtain the disruption nucleolus.

Problem 4

$$\begin{aligned} min \quad & r \\ s.t. \quad & d^1(S,x) \leq r \qquad \forall S \subset N (S \neq N, \emptyset) \\ & x(N) = v(N) \\ & x \geq 0 \end{aligned}$$

We can convert Problem 4 into the linear programming problem with the variable transformation of r. Hence, if we solve the resulting linear programming problem, we can get the disruption nucleolus.[18] If we cannot find a unique solution in Problem 4, we have to solve the next stage problems repeatedly until we obtain a unique solution.

2.3. *The Disruption Nucleolus (Charnes and Seiford)*

While Littlechild and Vaidya[1976] proposes a new propensity to disrupt, they refer to the difficulties of their propensity to disrupt. The following difficulties may arise when the core of the game is empty or the core of the game is not sufficiently large even if the core exists.[19]

- $d^1(S,x)$ may be infinite.
- $d^1(S,x)$ may be negative.
- $d^1(S,x)$ may be artificially positive.

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18) See Littlechild and Vaidya[1976](pp.154-156) as to the computational procedure of the disruption nucleolus.

19) See Littlechild and Vaidya[1976](pp.154).

Our model is represented by the convex game. It is well-known that the core exists and the range of core is relatively large in the convex game. So, we can ignore the above difficulties in our analysis.

Charnes and Seiford[1978] points out that the above difficulties arise from the fact that $d^1(S,x)$ is the ratio of the opportunity loss (7) and (8). Charnes and Seiford[1978] proposes the new propensity to disrupt in which the dissatisfaction of coalition S with allocation x is the weighted difference between (7) and (8). Their propensity to disrupt is represented by:

$$\alpha(N-S)\{x(N-S)-v(N-S)\} - \beta(S)\{x(S)-v(S)\} \tag{12}$$

The coefficients $\alpha(N-S)$ and $\beta(S)$ are suitably chosen weight or "normalization" factors.[20] Charnes and Seiford selects the reciprocal of the coalitional size as the weight. The propensity to disrupt in this case is :

$$d^2(S,x) = \frac{x(N-S)-v(N-S)}{n-s} - \frac{x(S)-v(S)}{s} \tag{13}$$

$n-s$ and s are the number of the coalition $N-S$ and S, *i.e.*, $|N-S|=n-s$ and $|S|=s$. We can easily see from (13) that the opportunity loss per person is computed for coalition $N-S$ and S, and the propensity to disrupt is the difference between the opportunity loss per person for $N-S$ and S. We can obtain (14) using the relationship $x(N-S)=v(N)-x(S)$ in (13).

$$d^2(S,x) = \left\{\frac{1}{n-s} + \frac{1}{s}\right\}\left\{\frac{s(v(N)-v(N-S))+(n-s)v(S)}{n} - x(S)\right\} \tag{14}$$

We define the first term in the second parenthesis of (14) as $w(S)$.

$$w(S) = \frac{s(v(N)-v(N-S))+(n-s)v(S)}{n} \tag{15}$$

$w(S)$ is a convex combination of $v(S)$ and $v(N)-v(N-S)$. It means that the value of $w(S)$ is in the interval between $v(S)$ and $v(N)-v(N-S)$. If we regard $w(S)$ as a game, the second parenthesis of (14) is $w(S)-x(S)$ and this is the excess of game $w(S)$. Therefore, we can interpret (14) as the dissatisfaction of coalition S with allocation x in game w.

As $v(N)-v(N-S) \geq v(S)$ in a convex game,[21] $w(S) \geq v(S)$. This means that players bargain with other players about the allocation based on the amount $w(S)$ that is greater or equal to $v(S)$.

If the allocation x is in the core, the following relationship holds.

20) See Charnes and Seiford[1978](p.38).

21) If a game is super-additive, $v(N) \geq v(N\text{-}S) + v(S)$.

$$v(S) \leq x(S) \leq v(N) - v(N - S) \tag{16}$$

(16) means that $v(N) - v(N - S)$ is the upper limit that coalition S can obtain. Hence, we can interpret the propensity to disrupt proposed by Charnes and Seiford[1978] as the dissatisfaction measure considering into the maximum amount that coalition S can obtain.

We can define the disruption nucleolus using (13). Since the procedure is the same as the disruption nucleolus of Littlechild and Vaidya[1976], we omit the definition of the disruption nucleolus of Charnes and Seiford[1974].

We can obtain the disruption nucleolus of Charnes and Seiford[1978] by solving a series of the linear programming problem. If we replace $d^1(S,x)$ in Problem 4 with $d^2(S,x)$, we can get the linear programming problems for the disruption nucleolus in this subsection. So, we also omit this problem here.

3. The Nucleolus Based on the Least Core

3.1. *The Least Core*

Young et al.[1980] extends the least core and proposes two nucleoli. One is the weak nucleolus and the other is the proportional nucleolus. Since these nucleoli are derived from the least core, we explain the least core briefly here.

We can obtain the least core by solving the following linear programming problem.

Problem 5

$$\begin{aligned} &min && \varepsilon \\ &s.t. && x(S) \geq v(S) - \varepsilon && \forall S \subset N \\ & && x(N) = v(\mathrm{N}) \\ & && x \geq 0 \end{aligned}$$

It is clear that Problem 5 is equivalent to Problem 1, which is the first stage problem for the nucleolus. We give another interpretation to Problem 5 by noticing the difference between the first constraints in Problem 1 and Problem 5.

The first constraint in Problem 1 defines the maximum excess. The right hand of the first constraint in Problem 5 is $v(\mathrm{S}) - \varepsilon$. When the core exists, it is clear that $\varepsilon \geq 0$ since $x(S) \geq v(S)$. If the value of ε is sufficiently large, the core does not exist. Hence, if we increase ε in $v(S) - \varepsilon$, it means narrowing the range of the core. Therefore, the optimal value of ε in Problem 5 is the upper limit for the existence of the core.

Problem 5 suggests the possibility that we can define the new nucleolus by changing the first constrains. If we replace ε in Problem 5 with the product of ε and the number of coalition S, we can define the weak nucleolus. If we replace ε in Problem 5 with the product of ε and the worth of coalition S, we can define the proportional nucleolus.

3.2. *The Weak Nucleolus*

The dissatisfaction measure in the nucleolus, the excess, is $e(S,x)=v(S)-x(S)$. The size of the coalition, the number of the coalition, is not considered in excess. We incorporate the size of the coalition into the dissatisfaction measure on the weak nucleolus. The dissatisfaction of coalition S with x is :

$$e^1(S,x) = \frac{1}{s}\{v(S)-x(S)\} \qquad \forall S \subset N(S \neq N,\emptyset) \tag{17}$$

(17) means that the excess is divided by the size of coalition S. We can obtain the weak nucleolus by solving the following linear programming problem.

Problem 6

$$\begin{aligned} &min \quad \varepsilon \\ &s.t. \quad x(S) \geq v(S) - s\cdot\varepsilon \qquad \forall S \subset N \\ &\qquad x(N) = v(\mathrm{N}) \\ &\qquad x \geq 0 \end{aligned}$$

Problem 6 is the first stage problem for the weak nucleolus. If the optimal solution in Problem 6 is not unique, we need to solve the next stage problems. These procedures continue until we get a unique solution.

3.3. *The Proportional Nucleolus*

The weak nucleolus is derived from dividing the excess by the number of the coalition. In the proportional nucleolus, we consider the dissatisfaction measure in which the excess is divided by the worth of the coalition $v(S)$. The dissatisfaction measure in the proportional nucleolus is defined by :

$$e^2(S,x) = \frac{1}{v(S)}\{v(S)-x(S)\} \qquad \forall S \subset N(S \neq N,\emptyset) \tag{18}$$

$v(S)$ is the value that coalition S can get by itself. This value grows as the service demand of the coalition increases in the convex game. So, we interpret $v(S)$ as the scale of the coalition. Hence, we regard (18) as the measure in which the scale of the coalition is considered.

If we adopt (18) as the dissatisfaction measure, there is a difficulty such that we cannot define $e^2(S,x)$ for $v(S)$=0. But this difficulty is not serious and we can avoid this difficulty by returning to the original problem, *i.e.*, Problem 5. (18) in an interpretation for our analysis. The constraint is $x(S) \geq v(S) - \varepsilon \cdot v(S)$ in the original problem. This constraint is equivalent to $x(S) \geq 0$ for $v(S)=0$.

The proportional nucleolus is computed by the following linear programming problem.

Problem 7

$$\begin{array}{lll} min & \varepsilon & \\ s.t. & x(S) \geq v(S) - \varepsilon \cdot v(S) & \forall S \subset N \\ & x(N) = v(N) & \\ & x \geq 0 & \end{array}$$

Problem 7 is the first stage problem. If we cannot find a unique solution in this problem, we have to solve the next stage problems until we get the unique solution.

4. The Analysis With Numerical Example

4.1. *Numerical Example*

We have described five cooperative game solutions that are similar to the nucleolus in Section 2 and 3. We examine the properties of these solutions using the numerical examples. We denote the numerical examples in our analysis here.

We use a 4-person game example. We suppose the following piece-wise linear cost function $C(q)$ here. It is clear that $C(q)$ is a concave function. Namely,

$$C(q) = \begin{cases} 40q & 0 \leq q < 15 \\ 20q+300 & 15 \leq q < 30 \\ 10q+600 & 30 \leq q \end{cases}$$

While a player or a coalition whose demand of the service is less than 15 cannot receive the benefit arising from the joint acquisition of the service, a player or a coalition whose demand of the service is more than 30 can achieve more cost saving. We analyze four cases described in Table 1.

q_i	Case 1	Case 2	Case 3	Case 4
q_1	5	5	5	15
q_2	10	10	20	20
q_3	20	12	25	25
q_4	25	30	30	30
Total	60	57	80	90

Table 1 : The Service Demand of the Departments

The cost of the service for coalition S is $C(S)=C(q(S))$ and this is depicted in Table 2.

$C(S)$	Case 1	Case 2	Case 3	Case 4
$C(\{1\})$	200	200	200	600
$C(\{2\})$	400	400	700	700
$C(\{3\})$	700	480	800	800
$C(\{4\})$	800	900	900	900
$C(\{1,2\})$	600	600	800	950
$C(\{1,3\})$	800	640	900	1,000
$C(\{1,4\})$	900	950	950	1,050
$C(\{2,3\})$	900	740	1,050	1,050
$C(\{2,4\})$	950	1,000	1,100	1,100
$C(\{3,4\})$	1,050	1,020	1,150	1,150
$C(\{1,2,3\})$	950	840	1,100	1,200
$C(\{1,2,4\})$	1,000	1,050	1,150	1,250
$C(\{1,3,4\})$	1,100	1,070	1,200	1,300
$C(\{2,3,4\})$	1,150	1,120	1,350	1,350
$C(\{1,2,3,4\})$	1,200	1,170	1,400	1,500

Table 2 : The Cost of the Service in Coalition S

We can estimate the characteristic function from Table 2 and (1). We omit $v(\{i\})$ in Table 3.because $v(\{i\})=0$ for all i in our example.

$v(S)$	Case 1	Case 2	Case 3	Case 4
$v(\{1,2\})$	0	0	100	350
$v(\{1,3\})$	100	40	100	400
$v(\{1,4\})$	100	150	150	450
$v(\{2,3\})$	200	140	450	450
$v(\{2,4\})$	250	300	500	500
$v(\{3,4\})$	450	360	550	550
$v(\{1,2,3\})$	350	240	600	900
$v(\{1,2,4\})$	400	450	650	950
$v(\{1,3,4\})$	600	510	700	1,000
$v(\{2,3,4\})$	750	660	1,050	1,050
$v(\{1,2,3,4\})$	900	810	1,200	1,500

Table 3 : Characteristic Function $v(S)$

Consider the capability of each player for cost saving from the data of Table 3. $v(N)-$

$v(N-\{i\})$ is the marginal increment by player i's participation in the coalition $N-\{i\}$. The maximum cost saving is achieved when the grand coalition N is formed in our example. So, we can regard $v(N)-v(N-\{i\})$ as the capability of player i.[22] The player whose value of $v(N)-v(N-\{i\})$ is high is in the advantageous position in the bargaining for the allocation of common cost and can expect a favorable allocation. The value of $v(N)-v(N-\{i\})$ in our four cases are summarized as Table 4.

Player	Case 1	Case 2	Case 3	Case 4
1	150	150	150	450
2	300	300	500	500
3	500	360	550	550
4	550	570	600	600

Table 4 : $v(N)-v(N-\{i\})$

The values of $v(N)-v(N-\{i\})$ in Table 4 are in the interval between 150 and 600. We refer to the player whose value of $v(N)-v(N-\{i\})$ is in the 100's, 300's, and 500's as the low contribution player, the middle contribution player, and the high contribution player, respectively.

There is one low contribution player, one middle contribution player, and two high contribution players in Case 1. This case corresponds to the situation where the departments have various demand of the service. Case 1 may be general in this sense.

There is one low contribution player, two middle contribution players, and one high contribution player in Case 2. While the departments have various demand of the service, this case is different from Case 1 in the number of the middle contribution and the high contribution players.

There is one low contribution player and three high contribution player in Case 3. The position of player 1 is extremely weak in the bargaining process for allocating common cost. So this may be the extreme case.

Case 4 describes the situation where the contribution of each player is not different extremely. So, the position of each player is almost the same in the bargaining process.

4.2. *Results of Our Examples*

The equal propensity to disrupt is the same as the disruption nucleolus of Littlechild and Vaidya[1976] in our four cases.[23] Hence, we will only show the results of Littlechild and

22) It is clear from (4) that $v(N)-v(N-\{i\})\geq v(S)-v(S-\{i\})$. So $v(N)-v(N-\{i\})$ is the maximum capacity of player i for cost saving.

23) As $d^1(S,x)=r$ for all $|S|=3$, the optimal value in Problem 4 is determined by $d^1(S,x)(|S|=3)$ and $x(N)=v(N)$. By $d^1(S,x)=1/d^1(N-S,x)$, $d_1(x)=d_2(x)=d_3(x)=d_4(x)$. It means that the equal propensity to disrupt is equal to the disruption nucleolus.

Vaidya[1976]. We summarize the nucleolus(*Nu*), the disruption nucleolus (Littlechild and Vaidya[1976]; *DN*1), the disruption nucleolus (Charnes and Seiford[1978]; *DN*2), the weak nucleolus (*WN*), and the proportional nucleolus(*PN*) as Table 5 to 8. We compute these solutions using *Mathematica*.[24] The results described below are the allocated amount of the cost saving. Refer to Appendix 2 as to the allocated cost.

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	75.00	90.00	168.75	37.50	0.00
2	150.00	180.00	206.25	137.50	100.00
3	312.50	300.00	256.25	337.50	366.67
4	362.50	330.00	268.75	387.50	433.33

Table 5 : Allocation of the Cost Saving in Case 1

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	75.00	88.04	153.75	37.50	0.00
2	150.00	176.09	191.25	147.50	121.50
3	187.50	211.30	206.25	207.50	202.50
4	397.50	334.57	258.75	417.50	486.00

Table 6 : Allocation of the Cost Saving in Case 2

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	75.00	100.00	225.00	37.50	0.00
2	325.00	333.33	312.50	337.50	338.46
3	375.00	366.67	325.00	387.50	400.00
4	425.00	400.00	337.50	437.50	461.54

Table 7 : Allocation of the Cost Saving in Case 3

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	300.00	321.43	356.25	300.00	288.46
2	350.00	357.14	368.75	350.00	346.15
3	400.00	392.86	381.25	400.00	403.85
4	450.00	428.57	393.75	450.00	461.54

Table 8 : Allocation of the Cost Saving in Case 4

4.3. *Examination of our Examples*

4.3.1 *The Nucleolus*

The nucleolus does not give the maximum or the minimum solution of the five solutions in every case. In other words, the nucleolus does not give the extreme solution. Hence, we

24) *Mathematica* is the trademark of Wolfram Research Corporation. See Appendix 1 as to the linear programming problems for the nucleolus in Case 1.

regard the nucleolus as the standard solution among five nucleoli.

We note the allocated amount of player 1 in Case 1,2, and 3 to clarify the properties of the nucleolus. Though the service demand of player 1 is the same in Case 1 and Case 2, $v(N)$ in Case 1 is larger than $v(N)$ in Case 2. But the allocated cost to player 1 remains the same in Case 1 and Case 2.

The service demand of player 1 is the same in Case 1 and Case 3. Though $v(N)$ in Case 3 is far larger than $v(N)$ in Case 1, the allocated amount to player 1 is unchanged. We should notice that the allocated amount player 2 is the same in Case 1 and Case 2 although $v(N)$ in Case 1 is larger than $v(N)$ in Case 2.

Player 1 is the low contribution player in Case 1,2, and 3. Though player 2 is the middle contribution player in Case 1 and 2, the contribution of player 2 is relatively lower than that of player 3 and 4 in Case 1 and 2.

We conclude from the above discussion that the nucleolus may not reflect the change of $v(N)$ in the allocation properly. Especially, we should pay attention to the case where the relatively low contribution players exist because there is a possibility that the low contribution player may not receive any benefit in the nucleolus allocation.

4.3.2. *The Disruption Nucleolus (Littlechild and Vaidya[1976];DN1)*

We can avoid the difficulty of the nucleolus by the use of *DN*1. We notice the allocation to player 1 in Case 1, 2, and 3. We can easily see that the allocated amount to player 1 decreases as $v(N)$ decreases and the allocated amount to player 1 increases as $v(N)$ increases.

The contribution of player 1 and 2 is lower than that of player 3 and 4 in Case 1. *DN*1 allocates more benefit to player 1 and 2 than the nucleolus in Case 1. Player 1, 2, and 3 are relatively low contribution player in Case 2. *DN*1 gives more benefit to these players than the nucleolus in Case 2. This applies to the allocation to player 1 in Case 3.

From the above analysis, we see that *DN*1 is likely to allocate more benefit to relatively low contribution players. If we want to reflect the change of the total cost saving in the allocation, *DN*1 is a promising solution.

4.3.3. *The Disruption Nucleolus (Charnes and Seiford[1978];DN2)*

We easily see that *DN*2 gives the equalized amount to all players in every cases. We notice the allocation to player 1 and 4 to clarify this.

Player 1 is the low contribution player and player 4 is the high contribution player in every case. *DN*2 provides us with the allocation in which the difference between the allocated amount of player 1 and that of player 4 is minimum in every case. Hence, the contribution of each player is not reflected sensitively in *DN*2. Player 1 is the only low contribution player in Case 3. No cost is allocated to player 1 and player 1 receives some subsidy from others in Case 3(See Table 11 in Appendix 2). As player 1 uses the service actually in this case, *DN*2 may give other players unfair feeling.

We should not use *DN*2 in case where extremely low contribution players exist. But *DN*2 is a desirable allocation scheme in the situation where common cost is allocated irrespective of the service usage.

4.3.4. *The Weak Nucleolus(WN)*

As the allocation to player 1 in Case 1, 2, and 3 is the same, *WN* has the similar property to the nucleolus. We note Case 1, 2, and 3 to clarify the difference between *WN* and the nucleolus.

Player 1 is the low contribution player and player 3 and 4 are the high contribution player in these cases. *WN* allocation to player 1 is the half of the nucleolus allocation in these cases. *WN* gives player 3 and 4 larger amount than the nucleolus allocation.

Player 2 is the relatively low contribution player in Case 1 and 2. *WN* gives player 2 smaller amount than the nucleolus in these cases. Player 2 is the high contribution player in Case 3. *WN* gives player 2 larger amount than the nucleolus in Case 3.

From the above discussion, *WN* is likely to allocate more amount to high contribution player and is likely to allocate less amount to the low contribution player. In other words, *WN* is more sensitive to player's contribution than the nucleolus.

Notice the allocated amount of player 2 in Case 1 and 2, we can find a difficulty of *WN*. $v(N)$ in Case 2 is smaller than $v(N)$ in Case 1. It means that the cost saving in Case 2 is smaller than the cost saving in Case 1. Furthermore, the service demand of player 2 remains the same in Case 1 and 2. But the allocated amount of player 2 in Case 2 is increased compared with Case 1. This denotes that *WN* allocation is likely to receive the changes of other player's service usage.

WN is the allocation scheme that is sensitive to the player's contribution. We should recognize that *WN* allocation is too sensitive to the contribution of the players in some cases when we use *WN* as an allocation scheme.

4.3.5. *The Proportional Nucleolus(PN)*

PN has the similar properties as *WN*. Namely, *PN* allocates more amount to the high contribution player and allocates less amount to the low contribution player. Hence, *PN* has the same difficulty as *WN*. If we notice the allocation to player 2 in Case 1 and 2, we can see that *PN* is more sensitive to the contribution of players than *WN*.

The *WN* allocation to player 1 in Case 1, 2, and 3 is zero. It means that *PN* may allocate no cost saving to the low contribution player in some cases. Hence, *PN* gives unfair feelings in the situation where extremely low contribution players exists.

PN is likely to receive the changes of the service demand drastically. For example, the difference of player 1's service demand between Case 1 and Case 4 is 10. While the allocated amount in Case 1 is zero, the allocated amount in Case 4 is 288.46.

We consider the case where player 1's service demand is 14 in Case 4. *PN* in this case is (254.65,547.24,405.12,462.99). Though the change of player 1's service damned is only one, the change of the allocated amount is 33.81. This difference is significant. But we cannot find extreme change of other player's allocated amount.

From the above analysis, we conclude that *PN* is similar to *WN* and is the most sensitive to player's contribution of five solutions examined here.

Conclusions

We have focused on the five cooperative game solutions, which are similar to the nucleolus, and examined the properties of these solutions using numerical examples. It should be noted that the results obtained in this article apply to the situations where the cost function of the service concave, *i.e.*, the scale of economy works.

We can indifferently uses these five solutions in the case where the service demand of each department is almost the same(Case 4). We should consider the followings when we select the allocation scheme in this case.

- What is the dissatisfaction measure of the department with the allocation?
- What is the desired properties of the allocation scheme?

If we can specify the above two points, the nucleoli examined here will give the satisfactory allocation to the users of the service in this case.

We encounter the some difficulties in the situation where the service demand of each departments is various. For example, the disruption nucleolus(Charnes and Seiford[1978]) in Case 3 allocate no cost to player 1 and subsidize player 1. Case 3 is the situation where player 1's demand of the service is extremely low. Though the nucleolus, the weak nucleolus, and the proportional nucleolus have desirable properties, they may not reflect the change of $v(S)$ in their allocation properly in some cases. As the disruption nucleolus(Littlechild and Vaidya[1976]) does not have serious difficulties in our model, this solution is promising allocation scheme in a convex game.

Recently, Thomas[1992] proposes MTPD(Minimum Total Propensity to Disrupt) as a common cost allocation scheme. This method consists of two steps. First, we solve the first stage problem for the nucleolus(Problem 1). Next, we solve the problem in which the total propensity to disrupt is minimized given the solution of the first stage problem.

If we use MTPD method, we can obtain the solution in the second iteration. Generally speaking, we have to solve the linear programming problems repeatedly to compute the nucleolus. The number of iterations is at most $n-1$ for n-person game. This applies to other nucleolus. Therefore, MTPD method is desirable in terms of the simplicity of the computation. The simplicity of the computation may be necessary in practice. But it is difficult to justify MTPD method theoretically because this method is a simple compromise of the nucleolus and the propensity to disrupt. Carter and Walker[1996] criticizes MTPD method for the lack of consistency.

When we apply the nucleoli examined here to practice, the complexity of the computational procedures is a problem. We have to resolve this problem to use these solutions in practice. For example, Franklin and Kochenberger[1994] computes MTPD solution using the spread sheet software. It may be necessary to devise a simple computational procedures for these cooperative game solutions.

If we apply these nucleoli to practice, we must specify the dissatisfaction measure in common cost allocation and examine the properties of this measure. The best approach to this problem is the field study or the questionnaire research. But these research are difficult, costly, and time consuming. Thomas[1988] uses a laboratory experiment to find the properties of the allocation schemes. His approach may be suggestive to our purpose.

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Appendix

Appendix 1 : The Linear Programming Problems for the Nucleolus

We only show the linear programming problems for the nucleolus in Case1. The computational procedures explained here apply to other solutions.

The first stage problem is:

$$
\begin{aligned}
&\min \quad r \\
&s.t. \\
&\quad x_i+r \geq 0 \qquad (i = 1,2,3,4) \\
&\quad x_1+x_2+r \geq 0 \\
&\quad x_1+x_3+r \geq 100 \\
&\quad x_1+x_4+r \geq 100 \\
&\quad x_2+x_3+r \geq 200 \\
&\quad x_2+x_4+r \geq 250 \\
&\quad x_3+x_4+r \geq 450 \\
&\quad x_1+x_2+x_3+r \geq 350 \\
&\quad x_1+x_2+x_4+r \geq 400 \\
&\quad x_1+x_3+x_4+r \geq 600 \\
&\quad x_2+x_3+x_4+r \geq 750 \\
&\quad x_1+x_2+x_3+x_4 = 900 \\
&\quad x_i \geq 0 \qquad (i = 1,2,3,4)
\end{aligned}
$$

The optimal value of the first stage problem is $r=-75$. The optimal solution of this problem is $x_1=75$, $x_2=225$, $x_3=425$, and $x_4=175$. Therefore, $B_1=\{\{1\},\{2,3,4\}\}$ and $X^1=\{(x_1, x_2, x_3, x_4) \mid x_1=75,\ x_2+x_3+x_4=825\}$.

The second stage problem is:

$$
\begin{aligned}
&\min \quad r \\
&s.t. \\
&\quad x_i+r \geq 0 \qquad (i = 2,3,4) \\
&\quad x_2+r \geq 0 \\
&\quad x_3+r \geq 100 \\
&\quad x_4+r \geq 100
\end{aligned}
$$

$$x_2+x_3+r \geq 275$$
$$x_2+x_4+r \geq 325$$
$$x_3+x_4+r \geq 525$$
$$x_2+x_3+r \geq 350$$
$$x_2+x_4+r \geq 400$$
$$x_3+x_4+r \geq 600$$
$$x_2+x_3+x_4 = 825$$
$$x_i \geq 0 \qquad (i = 2,3,4)$$

The optimal value of the second stage problem is $r=-75$. The optimal solution of this problem is x_2=150, x_3=350, and x_4=325. Therefore, B_2={{2}, {3,4}} and $X^2=\{(x_1,x_2, x_3, x_4) \mid x_1=75, x_2=150, x_3+x_4=675\}$.

The third stage problem is:

$$\min \quad r$$
s.t.
$$x_i+r \geq 150 \qquad (i=3,4)$$
$$x_3+r \geq 175$$
$$x_4+r \geq 175$$
$$x_3+r \geq 200$$
$$x_4+r \geq 250$$
$$x_3+x_4+r \geq 600$$
$$x_3+r \geq 275$$
$$x_4+r \geq 325$$
$$x_3+x_4 = 675$$
$$x_i \geq 0 \qquad (i = 3,4)$$

The optimal value of the second stage problem is $r=-37.5$. The optimal solution of this problem is x_3=312.5, and x_4=362.5. We can obtain the nucleolus in Case 1 from the above three problems.The nucleolus of this case is x=(75,150,312.5,362.5).

Appendix 2 : The Allocated Cost of our Examples

We can obtain the allocated cost to each department from the results of Table 5 to 8 using (2).

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	125.00	110.00	31.25	162.50	200.00
2	250.00	220.00	193.75	262.50	300.00
3	387.50	400.00	443.75	362.50	333.33
4	437.50	470.00	531.25	412.50	366.67

Table 9 : Allocated Cost in Case 1

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	125.00	111.96	46.25	162.50	200.00
2	250.00	233.91	208.75	252.50	278.50
3	292.50	268.70	273.75	272.50	277.50
4	502.50	565.43	641.25	482.50	414.00

Table 10 : Allocated Cost in Case 2

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	125.00	100.00	-25.00	162.50	200.00
2	375.00	366.67	387.50	362.50	361.54
3	425.00	433.33	475.00	412.50	400.00
4	475.00	500.00	562.50	462.50	438.46

Table 11 : Allocated Cost in Case 3

Player	*Nu*	*DN*1	*DN*2	*WN*	*PN*
1	300.00	278.57	243.75	300.00	311.54
2	350.00	342.86	331.25	350.00	353.85
3	400.00	407.14	418.75	400.00	396.15
4	450.00	471.43	506.25	450.00	438.46

Table 12 : Allocated Cost in Case 4