Middlemen in the Shapley-Shubik competitive markets for indivisible goods

Takayuki Oishi*, Shin Sakaue†

February 7, 2014

Abstract

We generalize the Shapley-Shubik market model for indivisible goods by considering the case where agents need middlemen to exchange their indivisible goods. In this model, there always exist competitive equilibria in which transaction takes place directly between sellers and buyers or indirectly through the middlemen. Furthermore, the incentives of middlemen to enter the market exist. We derive these results from the existence of an integral solution for a partitioning linear program.

Keywords: Middlemen; Competitive equilibrium; Partitioning linear program
JEL classification: C62; D50

1 Introduction

The seminal paper of Shapley and Shubik [1] analyzed a competitive market for indivisible goods, namely an assignment market. In the assignment market, each seller owns one unit of indivisible goods initially and she wants to sell it; and each buyer wants to purchase at most one unit of the indivisible goods. Shapley and Shubik showed that there always exists a competitive equilibrium which attains efficiency. Many economists have applied the assignment market model to analyses of housing markets and labor markets, e.g., Kaneko [2], and Kelso and Crawford [3].

In the real world, there are also various assignment markets with middlemen, e.g., housing markets with real estate brokers and intermediate labor markets. Introducing models different from the assignment market model,

*Assistant Professor, Aomori Public University, *E-mail addresses:*oishi@bb.nebuta.ac.jp
†Post-doctorial Fellow, Graduate School of Global Environmental Studies, Sophia University, *E-mail addresses:* shins@gs.econ.keio.ac.jp.

several researchers have investigated markets for indivisible goods with middlemen, e.g., Rubinstein and Wolinsky [4], Johri and Leach [5], Blume et al. [6], and Yano [7].[1]

Although the existing literature gives some insight into the study of markets with middlemen, the following important questions may be still left for us: When does a competitive equilibrium with middlemen exist? Do middlemen yield efficient allocations? Do the incentives of middlemen to enter the market exist?

Toward the purpose of our paper, we generalize the Shapley and Shubik's model by considering the case where agents need middlemen to exchange their indivisible goods. This situation can correspond to a lack of information on the both sides of the market or to inhibitive transaction costs. Here, search cost and matching cost are considered. Matching cost is regarded as opportunity cost for matching agents who search trading partners. Furthermore, middlemen are supposed to be matchmakers. The role of the middlemen is borrowed from Rubinstein and Wolinsky [4]. Middlemen can eliminate search costs of sellers and buyers by matching them, but the middlemen incur their matching costs. A profit of middlemen is interpreted as a brokerage fee. In addition, we assume that middlemen have the same matching skill, namely the matching cost is identical. Under this simple setting, it would be easy for us to understand how middlemen are related to the existence of a competitive equilibrium.

In this paper, we show that a competitive equilibrium with (resp. without) middlemen, which attains efficiency, always exists if (i) the sum of the matching costs of each seller and each buyer is relatively higher (resp. lower) than the matching cost of each middleman, and (ii) the number of middlemen (resp. marketplaces) is not less than the number of potential assignments of goods. The number of potential assignments is given by the minimum of the number of sellers (goods) and the number of buyers. We also show that the incentives of middlemen to enter the market always exist under the situation where the number of middlemen is equal to the number of the potential assignments of goods.

The results mentioned above are obtained by using a "partitioning linear program" proposed by Quint [8]. Using this linear program, Quint showed that the existence of an integral solution for the partitioning linear program guarantees emptiness of the core of multi-sided assignment games. On the other hand, our paper highlights an application of the partitioning linear program to an economic analysis of middlemen. By using the methodology in our

[1] Rubinstein and Wolinsky [4] and Johri and Leach [5] investigated the activity of middlemen who play a role of matchmakers from the viewpoint of search theory. Blume et al. [6] presented a trading network model in which each middleman is a market maker. Yano [7] incorporated outside competitive forces of middlemen in his market bargaining model in order to deal with a certain fairness in an M&A market.

paper, we can generalize our results easily in the context of middlemen who deal with multiple units of indivisible goods and who have different matching skill, respectively.[2] Therefore, our paper focuses on the simple model proposed here rather than a generalized model.

2 The model

We introduce an assignment market model with middlemen under a certain cost structure. Let $N_1 = \{i_1, i_2, \ldots, i_{n_1}\}$ and $N_3 = \{k_1, k_2, \ldots, k_{n_3}\}$ be the set of sellers and the set of buyers, respectively. We define the set of middlemen as $J^m = \{j_1^m, j_2^m, \ldots, j_{n_m}^m\}$, and the set of *dummy middlemen* as $J^d = \{j_1^d, j_2^d, \ldots, j_{n_d}^d\}$. A dummy middleman is interpreted as a *marketplace* in which a seller and a buyer trade each other and they incur *search costs*. A middleman plays a role of a matchmaker who incurs *matching costs*. Matching costs are opportunity costs for matching agents who search trading partners. Let N_2 be the set of all types of middlemen, namely $N_2 = J^m \cup J^d$ and $|N_2| = n_2 = n_m + n_d$. Note that $n_1, n_m, n_d, n_3 \in \mathbb{N}$.[3] Let $N = N_1 \cup N_2 \cup N_3$.

There are n_1 kinds of indivisible goods, and they are exchanged for money. Each seller $i \in N_1$ owns only one unit of indivisible goods initially, namely $\omega_i = 1$. Each middleman and each buyer own no unit of goods initially. Each middleman deals with at most one unit of goods between sellers and buyers. Each buyer consumes at most one unit of goods.

Assume that $n_d > \min\{n_1, n_3\}$. This means that the number of marketplaces is larger than the potential assignments of goods. In other words, there are many opportunities in which sellers and buyers meet each other.

Let us denote the demand and the supply of this market as follows. The notation we adopt is useful for formalizing the competitive equilibrium.

Sellers' side: Each seller i chooses one of the followings: (i) she sells one unit of her goods to a middleman in J^m; (ii) she sells it to a buyer through a dummy middleman (i.e. marketplace) in J^d, and she incurs her search cost $c_i \geq 0$; (iii) she consumes her goods by herself. Let x_i be the consumption of seller i, namely $x_i \in \{0, 1\}$.

Middlemen's side: Each middleman j in N_2 wants to sell at most one unit of goods to only a buyer. For this purpose, each middleman purchases at most one unit of goods from only a seller. Let the matching cost of each middleman $j \in N_2$ be given by $c_{j^m} \geq 0$ for each $j^m \in J^m$ and $c_{j^d} = 0$ for each $j^d \in J^d$. Let $\tilde{x}_{ij}$ be the supply of seller $i \in N_1$ to middleman $j \in N_2$, namely

[2] For the detail in the case of multiple trading by each middleman, see Oishi and Sakaue [9]. For the detail of an assignment market model with heterogeneous middlemen, see Oishi [10].

[3] We denote by $\mathbb{N}$ the set of natural numbers.

$\tilde{x}_{ij} \in \{0,1\}$. Note that each middleman $j \in N_2$ consumes no unit of goods which the middleman purchases from seller $i \in N_1$.

Buyers' side: Each buyer k chooses one of the followings in order to consume at most one unit of goods: (i) she purchases at most one unit of goods from only a middleman in J^m; (ii) she purchases at most one unit of goods from a seller through a dummy middleman in J^d, and she incurs her search cost $c_k \geq 0$. Let x_{ijk} be the consumption of buyer k, namely $x_{ijk} \in \{0,1\}$. In the case of $x_{ijk} = 1$, buyer $k \in N_3$ demands one unit of goods which middleman $j \in N_2$ purchases from seller $i \in N_1$. Moreover, let $\tilde{x}_{ijk}$ be the supply of middleman $j \in N_2$, namely $\tilde{x}_{ijk} \in \{0,1\}$. In the case of $\tilde{x}_{ijk} = 1$, middleman $j \in N_2$ supplies to buyer $k \in N_3$ one unit of goods which the middleman purchases from seller $i \in N_1$.

The set of *feasible allocations* of sellers, middlemen and buyers are given by conditions **A1**, **A2** and **A3**, respectively.

A1 : For all $i \in N_1,\ X_i \equiv \{(x_i, (\tilde{x}_{ij})_{j \in N_2}) \in \mathbb{Z}_+^{1+n_2} : x_i + \sum_{j \in N_2} \tilde{x}_{ij} = \omega_i = 1\}$.

A2 : For all $j \in N_2,\ X_j \equiv \{(\tilde{x}_{ijk})_{i \in N_1, k \in N_3} \in \mathbb{Z}_+^{n_1 n_3} : \sum_{i \in N_1} \sum_{k \in N_3} \tilde{x}_{ijk} \leq 1\}$.

A3 : For all $k \in N_3,\ X_k \equiv \{(x_{ijk})_{i \in N_1, j \in N_2} \in \mathbb{Z}_+^{n_1 n_2} : \sum_{i \in N_1} \sum_{j \in N_2} x_{ijk} \leq 1\}$.

Each seller and each buyer have utility functions on consumption, which are measured in terms of money. These functions of each seller and each buyer are given by $U_i : \mathbb{Z}_+ \to \mathbb{R}$ for all $i \in N_1$ and $U_k : \mathbb{Z}_+^{n_1 n_2} \to \mathbb{R}$ for all $k \in N_3$, respectively.[4] The functions $U_i(\cdot)$ and $U_k(\cdot)$ are non-decreasing. Assume $U_i(0) = 0$ and $U_k(\mathbf{0}) = 0$.[5] Let $\omega_i^j = (0, \ldots, 0, e_i^j, 0, \ldots, 0) \in \mathbb{Z}_+^{n_1 n_2}$, where $e_i^j = 1$. By e_i^j, we mean that middleman $j \in N_2$ exchanges one unit of goods ω_i. We denote by $U_k(\omega_i^j)$ the utility outcome of buyer k if she consumes seller i's goods through middleman $j \in N_2$.

For each $(i,k) \in N_1 \times N_3$, $m, m' \in J^m$ with $m \neq m'$, and $d, d' \in J^d$ with $d \neq d'$, $U_k(\omega_i^m) = U_k(\omega_i^{m'})$ and $U_k(\omega_i^d) = U_k(\omega_i^{d'})$. This assumption means that for an arbitrarily fixed $i \in N_1$ and an arbitrarily fixed $k \in N_3$ buyer k's utility is invariant whenever each middleman $j \in J^m$ (resp. each dummy middleman $j \in J^d$) matches seller i with buyer k. Under the assumption, middlemen have the same matching skill, namely the matching cost is identical. Let $c_{j^m} = c_m \geq 0$ for all $j^m \in J^m$, where c_m is constant.

Each middleman j in J^m purchases at most one unit of goods at a price $p_i^m \in \mathbb{R}_+$ from seller i. Also, middleman j in J^m sells at most one unit of i's initial goods at a price $q_i^m \in \mathbb{R}_+$ to buyer k. Similarly, each dummy middleman j in J^d purchases at most one unit of goods at a price $p_i^d \in \mathbb{R}_+$ from seller i. Also, dummy middleman j in J^d sells at most one unit of i's initial goods

[4] We denote by $\mathbb{Z}$ and $\mathbb{R}$ the set of integral numbers and the set of real numbers, respectively.

[5] $\mathbf{0} = (0, \ldots, 0)$ where $\mathbf{0} \in \mathbb{Z}_+^{n_1 n_2}$.

at a price $q_i^d \in \mathbb{R}_+$ to buyer k. We denote by $p = (p_i^m, p_i^d)_{i\in N_1} \in \mathbb{R}_+^{2n_1}$ and $q = (q_i^m, q_i^d)_{i\in N_1} \in \mathbb{R}_+^{2n_1}$ the lists of prices.

Utility outcomes of all agents are given by the followings.

- For all $i \in N_1$, $U_i(x_i) + (p_i^d - c_i)\sum_{j\in J^d}\tilde{x}_{ij} + p_i^m \sum_{j\in J^m}\tilde{x}_{ij}$.
- For all $j \in J^m$, $-\sum_{i\in N_1} p_i^m(\sum_{k\in N_3}\tilde{x}_{ijk}) + \sum_{i\in N_1}\sum_{k\in N_3}(q_i^m - c_j)\tilde{x}_{ijk}$.
- For all $j \in J^d$, $-\sum_{i\in N_1} p_i^d(\sum_{k\in N_3}\tilde{x}_{ijk}) + \sum_{i\in N_1}\sum_{k\in N_3} q_i^d\tilde{x}_{ijk}$.
- For all $k \in N_3$, $U_k((x_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1}\sum_{j\in J^d}(q_i^d + c_k)x_{ijk} - \sum_{i\in N_1}\sum_{j\in J^m} q_i^m x_{ijk}$.

The utility outcome of each seller $i \in N_1$ is interpreted as follows: This utility outcome consists of three components. The first component is seller i's utility from self-consumption of ω_i. The second component is seller i's net profit derived from that the seller sells ω_i to a buyer through a dummy middleman in J^d. In this situation, seller i incurs search cost c_i. The third component is seller i's revenue derived from that the seller sells ω_i to a middleman in J^m.

The utility outcome of each middleman $j \in N_2$ is interpreted as follows: This utility outcome consists of three components. The first component is the cost derived from that the middleman (resp. the dummy middleman) purchases one unit of goods from a seller. The second component is the revenue derived from that the middleman (resp. dummy middleman) sells one unit of goods to a buyer. The third component is matching costs. If a middleman is a dummy middleman, the third component is dropped.

The utility outcome of each buyer $k \in N_3$ is interpreted as follows: This utility outcome consists of three components. The first component is buyer k's utility from her consumption of ω_i through a middleman in N_2. The second component is buyer k's costs derived from her search and her payment to a seller if the buyer transacts with a seller directly. The third component is buyer k's costs derived from her payment to a seller if the buyer transacts with a seller indirectly through a middleman in J^m.

The following conditions **A4** and **A5** are the *market-clearing conditions.*

A4 : For all $(i,j) \in N_1 \times N_2$, $\sum_{k\in N_3}\tilde{x}_{ijk} = \tilde{x}_{ij}$.

A5 : For all $(i,j,k) \in N_1 \times N_2 \times N_3$, $x_{ijk} = \tilde{x}_{ijk}$.

A tuple $(\hat{p}, \hat{q}, \hat{x}) = ((\hat{p}_i^m, \hat{p}_i^d)_{i\in N_1}, (\hat{q}_i^m, \hat{q}_i^d)_{i\in N_1}, ((\hat{x}_i)_{i\in N_1}, (\hat{x}_{ijk})_{i\in N_1, j\in N_2, k\in N_3})) \in \mathbb{R}_+^{2n_1} \times \mathbb{R}_+^{2n_1} \times \mathbb{Z}_+^{n_1+n_1n_2n_3}$ is called a *competitive equilibrium* if $(\hat{p}, \hat{q}, \hat{x})$ satisfies

(I): for all $i \in N_1$, $U_i(\hat{x}_i) + \max\{\hat{p}_i^d - c_i, p_i^m\}(\omega_i - \hat{x}_i)$
$= \max_{(x_i, (\tilde{x}_{ij})_{j\in N_2})\in X_i}\left[U_i(x_i) + (p_i^d - c_i)\sum_{j\in J^d}\tilde{x}_{ij} + p_i^m\sum_{j\in J^m}\tilde{x}_{ij}\right]$,

(II): for all $j \in J^m$, $\sum_{i\in N_1}\sum_{k\in N_3}(\hat{q}_i^m - \hat{p}_i^m - c_j)\hat{x}_{ijk}$

$= \max_{(\tilde{x}_{ijk})_{i\in N_1, k\in N_3}\in X_j} \left[\sum_{i\in N_1}\sum_{k\in N_3}(\hat{q}_i^m - \hat{p}_i^m - c_j)\tilde{x}_{ijk}\right]$,

(III): for all $j \in J^d$, $\sum_{i\in N_1}\sum_{k\in N_3}(\hat{q}_i^d - \hat{p}_i^d)\hat{x}_{ijk}$

$= \max_{(\tilde{x}_{ijk})_{i\in N_1, k\in N_3}\in X_j} \left[\sum_{i\in N_1}\sum_{k\in N_3}(\hat{q}_i^d - \hat{p}_i^d)\tilde{x}_{ijk}\right]$,

(IV): for all $k \in N_3$, $U_k((\hat{x}_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1}\{(\hat{q}_i^d + c_k)\sum_{j\in J^d}\hat{x}_{ijk} + \hat{q}_i^m \sum_{j\in J^m}\hat{x}_{ijk}\}$

$= \max_{(x_{ijk})_{i\in N_1, j\in N_2}\in X_k} \left[U_k((x_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in N_1}\{(\hat{q}_i^d + c_k)\sum_{j\in J^d}x_{ijk} + \hat{q}_i^m\sum_{j\in J^m}x_{ijk}\}\right]$,

and

(V): for all $i \in N_1$, $\hat{x}_i + \sum_{j\in N_2}\sum_{k\in N_3}\hat{x}_{ijk} = \omega_i (= 1)$.

Condition (I), (II), (III), and (IV) are the *utility-maximizing conditions for each seller, each middleman, each dummy middleman, and each buyer*, respectively. These conditions are standard in economics. Condition (V) is equivalent to **A1** through **A5**, and it means the *equilibrium conditions for all goods.*

We call $(\hat{p}, \hat{q})$ a *competitive equilibrium price* if there exists a competitive equilibrium $(\hat{p}, \hat{q}, \hat{x})$. Then a *competitive outcome* is given by $(\hat{u}, \hat{v}, \hat{w}) \in \mathbb{R}^{n_1+n_2+n_3}$, where $\hat{u}_i$, $\hat{v}_j$, and $\hat{w}_k$ are the utility outcomes of seller $i \in N_1$, middleman $j \in N_2$, and buyer $k \in N_3$ in equilibrium $(\hat{p}, \hat{q}, \hat{x})$, respectively.

3 Results

We need a linear program in order to assert the results. Let π be the matching structure between the agents in N, namely $\pi \equiv \{\{i\}|i \in N\} \cup \{\{i,j,k\}|i \in N_1, j \in N_2, k \in N_3\}$. Let $y = (y_T)_{T\in\pi} \in \mathbb{R}^{|\pi|}$. Following Quint [11], let $a \equiv (a_T)_{T\in\pi} \in \mathbb{R}_+^{n+n_1n_2n_3}$ satisfying (i) $a_{\{i\}} = U_i(\omega_i)$ for all $i \in N_1$, (ii) $a_{\{j\}} = a_{\{k\}} = 0$ for all $j \in N_2$ and all $k \in N_3$, (iii) $a_{\{i,j,k\}} = U_k(\omega_i^j) - c_i - c_k$ for all $(i,j,k) \in N_1 \times J^d \times N_3$, and (iv) $a_{\{i,j,k\}} = U_k(\omega_i^j) - c_j$ for all $(i,j,k) \in N_1 \times J^m \times N_3$. Note that $a_{\{i,j,k\}}$ is the social surplus yielded by the transaction between seller $i \in N_1$ and buyer $k \in N_3$ through middleman $j \in N_2$. For the sake of simplicity, we assume $a_{\{i,j,k\}} \geq 0$ for all $(i,j,k) \in N_1 \times N_2 \times N_3$.[6]

[6]One may weaken this assumption, namely $a_{\{i,j,k\}} = \max\{U_k(\omega_i^j) - c_i - c_k, 0\}$ for all $(i,j,k) \in N_1 \times J^d \times N_3$ and $a_{\{i,j,k\}} = \max\{U_k(\omega_i^j) - c_j, 0\}$ for all $(i,j,k) \in N_1 \times J^m \times N_3$. Under this assumption, almost all of our results are invariant. In this paper, we adopt the simple assumption of $a_{\{i,j,k\}}$.

The partitioning linear program (Quint [8]), in short the PLP, is given by

$$(P) \quad : \quad \max_{(y_T)_{T\in\pi}} \sum_{T\in\pi} a_T y_T$$
$$s.t. \sum_{T\in\pi, T\ni i} y_T = 1 \ \text{ for all } i \in N$$
$$y_T \geq 0 \ \text{ for all } T \in \pi.$$

We denote by the PLP-III the PLP derived from the market with middlemen. We also denote by the PLP-II the PLP derived from the bilateral market between sellers and buyers. Formally, the PLP-II is given by the PLP under $\pi' \equiv \{\{i\}|i \in N_1 \cup N_3\} \cup \{\{i,k\}|i \in N_1, k \in N_3\}$, and under $a' \equiv (a'_T)_{T\in\pi'} \in \mathbb{R}_+^{n_1+n_3+n_1n_3}$ satisfying (i) $a'_{\{i\}} = a_{\{i\}}$ for all $i \in N_1$, and (ii) $a'_{\{i,k\}} = a_{\{i,j,k\}}$ for all $i \in N_1$, all $j \in N_2$ and all $k \in N_3$.[7]

Given an arbitrarily fixed pair $(i,k) \in N_1 \times N_3$, let $\alpha_{ik} \equiv U_k(\omega_i^m) - U_k(\omega_i^d)$ for all $(m,d) \in J^m \times J^d$. For the transaction between seller i and buyer k, α_{ik} is regarded as *gross benefit yielded by each middleman in* J^m when $\alpha_{ik} > 0$. On the other hand, for the transaction between seller i and buyer k, we can interpret $-\alpha_{ik}$ as *gross benefit yielded by each marketplace in* J^d when $\alpha_{ik} < 0$.

Next, we explain the established result by Quint [8]. This result plays an important role in the proof of our results. The dual problem of (P) is defined as

$$(D) \quad : \quad \min_{(b_i)_{i\in N}} \sum_{i\in N} b_i$$
$$s.t. \sum_{i\in T} b_i \geq a_T \ \text{ for all } T \in \pi.$$

The following theorem is proved by Quint [8].

Quint's Theorem *The set of optimal solutions to* (D) *is non empty if and only if* (P) *solves integrally (i.e. with all 0's and 1's).*

The first result shows that transaction always takes place indirectly between sellers and buyers through middlemen if (i) the net benefit yielded by each middleman is non-negative, and (ii) the number of middlemen is not less than the number of potential assignments of goods.

Proposition 1 *For all* $i \in N_1$ *and all* $k \in N_3$*, if* $\alpha_{ik} - c_m \geq -(c_i + c_k)$ *and* $n_m \geq \min\{n_1, n_3\}$*, there always exists a competitive equilibrium with middlemen.*

[7] The PLP-II is given by $\max_{(y_T)_{T\in\pi'}} \sum_{T\in\pi'} a'_T y_T$ subject to $\sum_{T\in\pi', T\ni i} y_T = 1$ for all $i \in N$ and $y_T \geq 0$ for all $T \in \pi'$.

Proof. **Step 1**: First, we show that the PLP-III always has an integral solution. According to Quint [8], the PLP-II always has an integral solution. Let $y' = (y'_T)_{T\in\pi'} \in \mathbb{R}^{n_1+n_3+n_1n_3}$ be an integral solution of the PLP-II. Let $\{(i'_s, k'_s)\}_{s=1}^m$ be all pairs $(i', k') \in N_1 \times N_3$ such that $y'_{\{i',k'\}} = 1$. Note that $0 \leq m \leq n_m$. Define $y^* = (y^*_T)_{T\in\pi} \in \mathbb{R}^{n+n_1n_2n_3}$ such that $y^*_{\{i'_s,j^m_s,k'_s\}} = y'_{\{i'_s,k'_s\}}$ for all $s = 1,\ldots,m$, $y^*_{\{i\}} = 1 - \sum_{k\in N_3} y'_{\{i,k\}}$ for all $i \in N_1$, $y^*_{\{j\}} = y^*_{\{k\}} = 0$ for all $j \in N_2$ and all $k \in N_3$, and $y^*_{\{i,j,k\}} = 0$ otherwise. Let $y = (y_T)_{T\in\pi} \in \mathbb{R}^{n+n_1n_2n_3}$ be a vector which satisfies the constraints of the PLP-III. Define $y'' = (y''_T)_{T\in\pi'} \in \mathbb{R}^{n_1+n_3+n_1n_3}$ such that $y''_{\{i,k\}} = \sum_{j\in N_2} y_{\{i,j,k\}}$ for all $i \in N_1$ and all $k \in N_3$, $y''_{\{i\}} = y_{\{i\}}$ for all $i \in N_1$. Since y' is an integral solution of the PLP-II and y'' satisfies the constraints of the PLP-II, $\sum_{T\in\pi} a_T y_T = \sum_{T\in\pi'} a'_T y''_T \leq \sum_{T\in\pi'} a'_T y'_T = \sum_{T\in\pi} a_T y^*_T$. Therefore y^* is an integral solution of the PLP-III.

Step 2: Let $\hat{y}$ be a vector in $\mathbb{Z}_+^{n_1+n_1n_2n_3}$ such that $\hat{y}_i \equiv y^*_{\{i\}}$ for all $i \in N_1$ and $\hat{y}_{ijk} \equiv y^*_{\{i,j,k\}}$ for all $\{i,j,k\} \in \pi$. Let S_P be the set of all $\hat{y}$. Let C be given by the set of utility vectors $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^n$ satisfying (1) *the π-partition efficiency conditions*: for $\hat{y} \in \{0,1\}^{n_1+n_1n_2n_3}$, $\bar{u}_i + \bar{v}_j + \bar{w}_k = a_{\{i,j,k\}}$ if $\hat{y}_{ijk} = 1$; $\bar{u}_i = a_{\{i\}}$ if $\hat{y}_i = 1$; (2) *the stability conditions*: $\bar{u}_i + \bar{v}_j + \bar{w}_k \geq a_{\{i,j,k\}}$ if $\{i,j,k\} \in \pi$; $\bar{u}_i \geq a_{\{i\}}$ if $i \in N_1$; $\bar{v}_j \geq a_{\{j\}} = 0$ if $j \in N_2$; $\bar{w}_k \geq a_{\{k\}} = 0$ if $k \in N_3$. Using the Quint's theorem and the *complementary slackness condition*, we will show that the set C is not empty. Let us recall the complementary slackness condition (Dantzig [12], pp.135-136): Let y be a vector which satisfies the constraints of (P). Let b be a vector which satisfies the constraints of (D). Then y is a solution of (P) and b is a solution of (D) if and only if $\sum_{T\in\pi} y_T(\sum_{i\in T} b_i - a_T) = 0$. There exists a $\hat{y}$, which is derived from an integral solution y^* for (P). By the Quint's theorem, the set of optimal solutions to (D) is nonempty. Fix an arbitrary $(u', v', w') \in D$. It is sufficient to show that (u', v', w') satisfies (i) $u'_i + v'_j + w'_k = a_{\{i,j,k\}}$ if $\hat{y}_{ijk} = 1$ and (ii) $u'_i = a_{\{i\}}$ if $\hat{y}_i = 1$. By the complementary slackness condition, $u'_i + v'_j + w'_k = a_{\{i,j,k\}}$ if $\hat{y}_{ijk} = 1$ and $u'_i = a_{\{i\}}$ if $\hat{y}_i = 1$, which completes the proof of the claim.

Step 3: Let $\hat{y} \in S_P$. Then we have that $\bar{v}_j = 0$ for $j \in N_2$ such that $\hat{y}_{ijk} = 0$ for all $i \in N_1$ and all $k \in N_3$. Furthermore, $\bar{v}_m = \bar{v}_{m'}$ for $m, m' \in J^m$ with $m \neq m'$, and $\bar{v}_d = \bar{v}_{d'}$ for $d, d' \in J^d$ with $d \neq d'$. The proof is omitted since it is matter of simple calculation.

Next, fix an arbitrary $(\bar{u}, \bar{v}, \bar{w}) \in C$. We set a price list $(\hat{p}, \hat{q})$ satisfying $\hat{p}^m_i = \bar{u}_i$ for all $i \in N_1$, $\hat{p}^d_i = \bar{u}_i + c_i$ for all $i \in N_1$, $\hat{q}^m_i = \bar{u}_i + \bar{v}_{j^m} + c_m$ for all $(i, j^m) \in N_1 \times J^m$ and $\hat{q}^d_i = \bar{u}_i + \bar{v}_{j^d} + c_i$ for all $(i, j^d) \in N_1 \times J^d$. Note that $\hat{q}$ is well-defined by Step 3.

Step 4: The rest of the proof is to show that $(\hat{p}, \hat{q}, \hat{y})$ is a competitive equilibrium yielding the competitive outcome $(\bar{u}, \bar{v}, \bar{w})$. Let $\hat{y}_i = x^*_i$, $\hat{y}_{ijk} = x^*_{ijk} = \tilde{x}^*_{ijk}$ and $\sum_{k\in N_3} \hat{y}_{ijk} = \tilde{x}^*_{ij}$. Note that **A1**, **A2** and **A3** are satisfied. The basic line of our proof is almost the same as Shapley and Shubik [1]. We will discuss the different part from the proof of Shapley and Shubik.

Substep 4.1: Fix an arbitrary middleman $j^m \in J^m$. We will show that middleman j^m obtains $\hat{q}_i^m - \hat{p}_i^m - c_m$ or 0.

Case 1: $\tilde{x}_{ijk}^* = 0$ for all $i \in N_1$ and all $k \in N_3$. In this case, $0 \geq \hat{q}_i^m - \hat{p}_i^m - c_m$ must be satisfied. We can check it since $0 = \bar{v}_j = (\bar{u}_i + \bar{v}_j) - \bar{u}_i = \hat{q}_i^m - \hat{p}_i^m - c_m$.

Case 2: There exists a pair $(i,k) \in N_1 \times N_3$ such that $\tilde{x}_{ij^mk}^* = 1$ and $\tilde{x}_{i'j^mk'}^* = 0$ for all $(i',k') \in N_1 \times N_3$ with $(i',k') \neq (i,k)$. In this case, $\hat{q}_i^m - \hat{p}_i^m - c_m \geq \hat{q}_{i'}^m - \hat{p}_{i'}^m - c_m$ for all $i' \in N_1 \setminus \{i\}$, and $\hat{q}_i^m - \hat{p}_i^m - c_m \geq 0$ must be satisfied. We can check it since $\hat{q}_i^m - \hat{p}_i^m - c_m = (\bar{u}_i + \bar{v}_{j^m}) - \bar{u}_i = \bar{v}_{j^m} = (\bar{u}_{i'} + \bar{v}_{j^m}) - \bar{u}_{i'} = \hat{q}_{i'}^m - \hat{p}_{i'}^m - c_m$, and $\bar{v}_{j^m} \geq 0$.

Substep 4.2: Fix an arbitrary middleman $j^d \in J^d$. By Step 3, middleman j^d always obtains 0. Therefore, $\hat{q}_i^d - \hat{p}_i^d = \bar{v}_{j^d} = 0$ for all $i \in N_1$. ∎

Proposition 2 *A competitive equilibrium with middlemen is Pareto efficient.*

Proof. We show that each competitive outcome of the market belongs to the core of the induced assignment game. In the induced game, one considers a restricted set of feasible coalitions, the ones of size 1 (seller not selling, middleman not matching, or buyer not buying) and coalitions $\{i,j,k\}$ of size 3 that coincide with a seller i selling one unit of goods to a middleman j who sells himself the goods to k.

Let $(\hat{p}, \hat{q}, \hat{x})$ be a competitive equilibrium. One can check straightforwardly that the competitive outcome $(\hat{u}, \hat{v}, \hat{w})$ is individually rational, that is, $\hat{u}_i \geq U_i(\omega_i)$, $\hat{v}_j \geq 0$, $\hat{w}_k \geq U_k(\mathbf{0}) = 0$ for all $i \in N_1$, all $j \in N_2$ and all $k \in N_3$. If the outcome $(\hat{u}, \hat{v}, \hat{w})$ is blocked, it must be the case that there exists $(i,j,k) \in N_1 \times J^m \times N_3$ such that

$$\hat{u}_i + \hat{v}_j + \hat{w}_k < a_{\{i,j,k\}} = U_k(x_{ijk}) - c_m.$$

It holds

$$U_k(x_{ijk}) - c_m = p_i^m + (-p_i^m + q_i^m - c_m) + (U_k(x_{ijk}) - q_i^m).$$

From the definition of equilibrium, it must be the case that $\hat{u}_i \geq p_i^m$, $\hat{v}_j \geq -p_i^m + q_i^m - c_m$, and $\hat{w}_k \geq U_k(x_{ijk}) - q_i^m$, which is a contradiction. ∎

Corollary 1 *The set of competitive equilibria with middlemen coincides with the core.*

Proof. By Step 2 in the proof of Proposition 1, it is obvious that the core belongs to the set of competitive equilibria with middlemen. By the proof of Proposition 2, the set of competitive equilibria with middlemen belongs to the core. ∎

In contrast, the following result shows that transaction always takes place directly between sellers and buyers if the net benefit yielded by each marketplace is non-negative.

Proposition 3 *For all* $i \in N_1$ *and all* $k \in N_3$*, if* $-\alpha_{ik} - (c_i + c_k) \geq -c_m$*, there always exists a competitive equilibrium without middlemen.*

Proof. It is sufficient to show that if $c_i + c_k \leq -\alpha_{ik}$ for all $i \in N_1$ and all $k \in N_3$ then the competitive outcome of each dummy middleman is zero. Assume $n_d \geq \min\{n_1, n_3\}$. It is clear that there always exists a competitive equilibrium of the market with dummy middlemen, since the proof is the same as Proposition 1. Next, we show $\bar{v}_j = 0$ for all $j \in J^d$ if $n_d > \min\{n_1, n_3\}$. Let y^* be an integral solution of the PLP-III. Let $\hat{y}^*$ be a vector in S_P such that $\hat{y}^*_i \equiv y^*_{\{i\}}$ for all $i \in N_1$ and $\hat{y}^*_{ijk} \equiv y^*_{\{i,j,k\}}$ for all $\{i, j, k\} \in \pi$. Using the constraints of the PLP-III, we have $\sum_{i \in N_1} \sum_{j \in N_2} \sum_{k \in N_3} \hat{y}^*_{ijk} = \sum_{i \in N_1} \sum_{j \in N_2} \sum_{k \in N_3} y^*_{\{i,j,k\}} = \sum_{i \in N_1} (1 - y^*_{\{i\}}) = n_1 - \sum_{i \in N_1} y^*_{\{i\}} \leq n_1$. Similarly, we have $\sum_{k \in N_3} \sum_{i \in N_1} \sum_{j \in N_2} \hat{y}^*_{ijk} = n_3 - \sum_{k \in N_3} y^*_{\{k\}} \leq n_3$. Thus, $\sum_{i \in N_1} \sum_{j \in N_2} \sum_{k \in N_3} \hat{y}^*_{ijk} \leq \min\{n_1,\, n_3\} < n_d$. Suppose that for all $j \in J^d$ there exists a pair $(i, k) \in N_1 \times N_3$ such that $\hat{y}^*_{ijk} = 1$. Then $\sum_{j \in N_2} \sum_{i \in N_1} \sum_{k \in N_3} \hat{y}^*_{ijk} = \sum_{j \in J^d} \sum_{i \in N_1} \sum_{k \in N_3} \hat{y}^*_{ijk} \geq \sum_{j \in J^d} 1 = n_d$, which is a contradiction. Therefore there exists $j \in J^d$ such that $\hat{y}^*_{i'jk'} = 0$ for all $(i', k') \in N_1 \times N_3$, which implies $\bar{v}_j = 0$ for all $j \in J^d$ because of Step 3 in the proof of Proposition 1. ■

Remark 1 *As shown in Shapley and Shubik [1], the set of competitive equilibria without middlemen coincides with the core.*

Corollary 2 *For all* $i \in N_1$ *and all* $k \in N_3$*, if* $\alpha_{ik} - c_m \geq -(c_i + c_k)$ *and* $n_m > \min\{n_1, n_3\}$*, the competitive outcome of each middleman in* J^m *is zero.*

Proof. It is obvious by Proposition 1 and the same argument as Proposition 3. ■

Finally, we show that the incentives of middlemen to enter the market always exist under the situation where the number of middlemen is equal to the number of the potential assignments of goods.

Proposition 4 *For all* $i \in N_1$*, all* $m \in J^m$*, and all* $k \in N_3$*, if*

$$\alpha_{ik} - c_m > -(c_i + c_k),\ U_k(\omega_i^m) - c_m > U_i(\omega_i),\ \text{and } n_m = \min\{n_1, n_3\},$$

there always exists a competitive equilibrium with middlemen who gain positive utility outcomes.

Proof. By Proposition 1, there exist $\hat{y} \in S_P$, and $(\bar{u}, \bar{v}, \bar{w}) \in C$, which yields a competitive equilibrium $(\hat{p}, \hat{q}, x^*)$, where $x^* = ((x_i^*)_{i \in N_1}, (x_{ijk}^*)_{i \in N_1, j \in N_2, k \in N_3})$. By $\alpha_{ik} - c_m > -(c_i + c_k)$, $n_m = \min\{n_1, n_3\}$ and Proposition 1, for all $m \in J^m$ $\sum_i \sum_k x_{imk}^* = 1$, and for all $d \in J^d$ $\sum_i \sum_k x_{idk}^* = 0$. By Step 3 in the proof of Proposition 1, for $m, m' \in J^m$ with $m \neq m'$ $\bar{v}_m = \bar{v}_{m'}$, and for $d, d' \in J^d$ with $d \neq d'$ $\bar{v}_d = \bar{v}_{d'} = 0$. If $\bar{v}_m > 0$ for all $m \in J^m$, there is nothing to prove.

In the following argument, let $\bar{v}_m = 0$ for all $m \in J^m$. Fix an arbitrary $i \in N_1$ and an arbitrary $k \in N_3$. Since $U_k(\omega_i^m) - c_m > U_i(\omega_i)$, it is sufficient to consider the case where for all $m' \in J^m$ such that $x_{im'k}^* = 1$, $\bar{u}_i > U_i(\omega_i)$ or $\bar{w}_k > 0$. Let $\hat{u}_i = \bar{u}_i - \epsilon$ and $\hat{v}_m = \bar{v}_m + \epsilon$ for enough small $\epsilon > 0$ if $x_{imk}^* = 1$ and $\bar{u}_i > U_i(\omega_i)$. Let $\hat{w}_k = \bar{w}_k - \epsilon$ and $\hat{v}_m = \bar{v}_m + \epsilon$ for enough small $\epsilon > 0$ if $x_{imk}^* = 1$, $\bar{u}_i = U_i(\omega_i)$ and $\bar{w}_i > 0$. Otherwise, $\hat{u}_i = \bar{u}_i$, $\hat{v}_j = \bar{v}_j$, and $\hat{w}_k = \bar{w}_k$ for all $(i, j, k) \in \{(i, j, k) \in N_1 \times N_2 \times N_3 : x_{ijk}^* = 0\}$. Furthermore, we must show the following claim: for all $(i, d, k) \in N_1 \times N_2 \times N_3$ $\bar{u}_i + \bar{v}_d + \bar{w}_k > a_{\{i,d,k\}}$. Since $\alpha_{ik} - c_m > -(c_i + c_k)$, $U_k(\omega_i^m) - c_m > U_k(\omega_i^d) - c_i - c_k$, which implies $a_{\{i,m,k\}} > a_{\{i,d,k\}}$. By Step 3 in the proof of Proposition 1, $\bar{v}_d = 0$. Since $(\bar{u}, \bar{v}, \bar{w}) \in C$, $\bar{u}_i + \bar{v}_m + \bar{w}_k = \bar{u}_i + \bar{w}_k \geq a_{\{i,m,k\}}$. Thus, $\bar{u}_i + \bar{v}_d + \bar{w}_k = \bar{u}_i + \bar{w}_k \geq a_{\{i,m,k\}} > a_{\{i,d,k\}}$, which is the desired claim. Therefore, $(\hat{u}, \hat{v}, \hat{w}) \in C$, which yields the competitive equilibrium $(\hat{p}, \hat{q}, x^*)$. ■

The following example illustrates that the competitive outcome of each middleman is not necessarily zero.

Example 3 *Let* $N_1 = \{i_1, i_2, i_3\}$, $N_3 = \{k_1, k_2, k_3\}$ *and* $N_2 = J^m \cup J^d$, *where* $J^m = \{j_1^m, j_2^m, j_3^m\}$ *and* $J^d = \{j_1^d, j_2^d, j_3^d\}$. *Let the utility functions of all agents be given by (i)* $U_i(\omega_i) = 30$ *for all* $i \in N_1$; *(ii)* $U_{k_1}(\omega_{i_1}^m) = 120$ *for all* $m \in J^m$ *; (iii)* $U_{k_2}(\omega_{i_2}^m) = 150$ *for all* $m \in J^m$; *(iv)* $U_{k_3}(\omega_{i_3}^m) = 180$ *for all* $m \in J^m$; *(v)* $U_k(\omega_i^j) = 60$, *otherwise. Let the costs of all agents be given by (i)* $c_i = 10$ *for all* $i \in N_1$; *(ii)* $c_m = 20$; *(iii)* $c_k = 10$ *for all* $k \in N_3$. *There exists a competitive equilibrium price* $(\hat{p}, \hat{q})$ *such that (i)* $\hat{p}_{i_1}^m = 30$, $\hat{p}_{i_2}^m = 40$, *and* $\hat{p}_{i_3}^m = 50$; *(ii)* $\hat{p}_{i_1}^d = 40$, $\hat{p}_{i_2}^d = 50$, *and* $\hat{p}_{i_3}^d = 60$; *(iii)* $\hat{q}_{i_1}^m = 100$, $\hat{q}_{i_2}^m = 110$, *and* $\hat{q}_{i_3}^m = 120$; *(iv)* $\hat{q}_{i_1}^d = 40$, $\hat{q}_{i_2}^d = 50$, *and* $\hat{q}_{i_3}^d = 60$. *Given this equilibrium price, we have competitive outcome* $(\hat{u}, \hat{v}, \hat{w})$ *satisfying (i)* $\hat{u}_{i_1} = 30$, $\hat{u}_{i_2} = 40$, $\hat{u}_{i_3} = 50$; *(ii)* $\hat{v}_m = 50$ *for all* $m \in J^m$, *and* $\hat{v}_d = 0$ *for all* $d \in J^d$; *(iii)* $\hat{w}_{k_1} = 20$, $\hat{w}_{k_2} = 40$ *and* $\hat{w}_{k_3} = 60$.

Acknowledgements

One of the authors (T. Oishi) is financially supported by Grant-in-Aid for Young Scientists (B), Project #25870575. We would like to thank Mamoru Kaneko, Manabu Toda, and Makoto Yano for their helpful suggestions on previous versions of our paper. These versions have been presented at the second Keio/Kyoto International Conference on Market Quality Economics, the 2010 Japanese Economic Association Annual Spring Meeting at Chiba University, and the 17th Decentralization Conference at University of Tsukuba. Finally, we would also thank Naoto Aoyama and Jun Tomioka for their comments on our paper.

References

[1] Shapley LS, Shubik M. The assignment game I: the core. Int J Game Theory 1971, 1, 111-30.

[2] Kaneko M. Housing market with indivisibilities. J. Urban Econ 1983, 13, 22-50.

[3] Kelso AS, Crawford VP. Job matching coalition formation and gross substitutes. Econometrica 1982, 50, 1483-504.

[4] Rubinstein A, Wolinsky A. Middlemen. Q J Econ 1987, 102, 581-93.

[5] Johri A, Leach J. Middlemen and the allocation of heterogeneous goods. Int Econ Rev 2002, 43, 347-61.

[6] Blume LE, Easley D, Kleinberg J, Tardos E. Trading networks with price-setting agents. Games Econ Behav 2009, 67, 36-50.

[7] Yano M. Competitive fairness and the concept of a fair price under Delaware Law on M&A. Int J Econ Theory 2008, 4, 175-90.

[8] Quint T. Necessary and sufficient conditions for balancedness in partitioning games. Math Soc Sci 1991, 22, 87-91.

[9] Oishi T, Sakaue S. Competitive equilibria in a market for indivisible commodities with middlemen. Keio University/Kyoto University Global COE Discussion Paper Series 2009, DP2008-016.

[10] Oishi T. Distributive function of informal and formal economic systems: a cooperative game theoretic analysis. Ph.D. dissertation, Keio University, 2011.

[11] Quint T. The core of an m-sided assignment game. Game Econ Behav 1991, 3, 487-503.

[12] Dantzig GB. Linear programming and extensions. Princeton, NJ, USA, Princeton University Press, 1963.