Duality and anti-duality for cooperative game theory with economic applications

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Abstract

We demonstrate that the notions of duality and anti-duality are useful for analyzing several properties for single-valued solutions for coalitional games with economic applications. First, we propose a new monotonic property derived from the anti-dual of "population monotonicity". Using the notion of anti-duality, we derive sufficient conditions under which the new monotonic property is satisfied by a single-valued solution on the domain of convex games. Next, using the notion of duality and axioms involved in several axiomatizations of the Shapley rule for airport problems, we axiomatize the Shapley rule for liability problems, bidding ring problems, and polluted river problems. Finally, using the notion of anti-duality, we uncover the hidden relationship between the nucleolus rules for claims problems and for public good problems.

Keywords: duality; anti-duality; population monotonicity; allocation problems; axiomatization; the Shapley value; nucleolus

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1 Introduction

In "claims problems", Thomson and Yeh (2008) introduced operators on the space of division rules and uncovered the underlying structure of the space of division rules. The notion of "duality" plays an important role in their analysis.

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For each claims problem, this notion gives us the dual of the problem. Also, the notion of duality is applied to division rules: A division rule is said to be "self-dual", if the outcome this division rule chooses for each claims problem and the outcome the same division rule chooses for the dual of each claims problem always coincide with each other.

Analogously to claims problems, one can define "dual games" and "dual solutions" in cooperative game theory. These concepts can help us to uncover the hidden structure of solutions, axioms, and axiomatizations on the domain of all coalitional games with transferable utility (TU games, for short) (Funaki 1998). However, the notion of duality for TU games has weakness: First, particular domains of TU games (for instance, the classes of balanced games and convex games) are not closed under the duality operator. Second, although the Shapley value is a self-dual solution on the domain of all TU games, there are few self-dual solutions.

Oishi et al. (2013) proposed the notion of "anti-duality" for TU games. Given a TU game v, the "anti-dual" of the game, introduced by Oishi and Nakayama (2009), is defined as the dual of $-v$. One can also define the notion of a "self-anti-dual solution". Unlike the notion of duality, the notion of anti-duality has the following advantages: Important classes of games, such as the class of balanced games and the class of convex games, are closed under the anti-duality operator. Also, there are many self-anti-dual solutions on the class of all TU games, on the class of balanced games, and on the class of convex games.[1] Taking advantage of these facts, Oishi et al. (2013) axiomatized the core (Gillies 1959) on the domain of balanced games, the Shapley value (Shapley 1953) on the domain of all TU games, and the Dutta-Ray solution (Dutta and Ray 1989) on the domain of convex games.

The purpose of this paper is to demonstrate that the notions of duality and anti-duality are useful in different contexts. We follow three strategies.

First, we derive a new monotonic property as the anti-dual of "population monotonicity". Population monotonicity (Thomson 1983) states that if new agents arrive, the payoffs to agents that are present initially have to increase. The monotonicity property we propose says that if the contribution of agents to a particular coalition to which they do not belong is the coalitional worth for the agents in a new game, then they weakly gain in this game. We refer to it as "coalitional contribution monotonicity". Hokari and Gellekom (2002) provided sufficient conditions for a single-valued solution to be population monotonic on the domain of convex games. Using anti-duality, we derive sufficient conditions for a single-valued solution to be coalitional contribution monotonic on the

[1] For instance, on the domain of all TU games, the Shapley value, the prenucleolus, the modified nucleolus, and the prekernel are sefl-anti-dual solutions; on the domain of balanced games, the core, the Shapley value, the nucleolus, the modified nucleolus, and the prekernel are self-anti-dual solutions; and on the domain of convex game, the core, the Shapley value, the nucleolus, and the Dutta-Ray solution are self-anti-dual solutions (see Oishi et al. 2013).

domain of convex games.

Next, we apply the notion of duality to “airport problems”, “bidding ring problems”, “liability problems”, and “polluted river problems”. Airport problems are cost sharing problems of an airstrip among airlines (Littlechild and Owen 1973; Thomson 2007, for a survey of the literature). A bidding ring problem (Graham et al. 1990) describes the situation where bidders form a ring in a single-object English auction. The ring reduces or eliminates buyer competition, thereby securing an advantage over the seller. The problem forced by the members of the ring is to share the benefit of their strategy. A liability problem (Dehez and Ferey 2013) describes the situation where someone suffers a cumulative injury that is caused by several persons in succession. Each injuring party has taken a wrongful act after his predecessor's wrongful act. A wrongful act taken by the first injuring party is the root of the injury. The injured party is entitled to compensation. This problem is to determine how the injuring parties should share the compensation. A polluted river problem (Ni and Wang 2007) describes the situation where states are located along a river and each state produces some pollutants. Each state is responsible for cleaning not only its own watercourse but also all downstream watercourses. The problem is to determine how the states should share the total cleaning cost. Fragnelli and Marina (2010) and Chun et al. (2012) axiomatized the Shapley rule for airport problems. Considering these axioms in a duality relation, we axiomatize the Shapley rule for liability problems, bidding ring problems, and polluted river problems.

Finally, we apply the notion of anti-duality to “claims problems”, and a certain class of “public good problems”. Claims problems deal with the situation where the liquidation value of a bankrupt firm has to be allocated between its creditors, but there is not enough to honor the claims of all creditors. The problem is to determine how the creditors should share the liquidation value (O'Neill 1982; Thomson 2003, for a survey of the literature). Public good problems deal with the situation where a fixed size of a public good can be provided at a cost, and each agent consumes the public good. The social benefit is the difference between the sum of benefits of all agents and the cost of the public good. The problem is to determine how the agents should share the social benefit. Using anti-duality, we uncover the hidden relationship between the nucleolus rules (Schmeidler 1969) for claims problems and for public good problems.

The rest of this paper is organized as follows. Section 2 explains the notions of duality and anti-duality for cooperative game theory. In Section 3, we introduce the anti-dual of population monotonicity. On the domain of convex games, we derive sufficient conditions under which this property is satisfied by a single-valued solution. In Section 4, using duality, we axiomatize the Shapley rule for bidding ring problems, liability problems, and polluted river problems.

Finally, using anti-duality, we analyze the relationship between the nucleolus rules for claims problems and for public good problems.

2 Preliminaries

We introduce the notions of duality and anti-duality for solutions and axioms of cooperative game theory. There is a universe of potential agents, denoted $\mathcal{I} \subseteq \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.[2] Let $\mathcal{N}$ be the class of non-empty and finite subsets of $\mathcal{I}$, and $N \in \mathcal{N}$. A **coalitional game with transferable utility for $\boldsymbol{N}$** (a **TU game for $\boldsymbol{N}$**, for short) is a function $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. For all $S \in 2^N$, $v(S)$ represents what coalition S can achieve on its own. Let $\boldsymbol{\mathcal{V}^N}$ be the **class of TU games for $\boldsymbol{N}$**, and $\mathcal{V} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{V}^N$.

Given a TU game v for N and $N' \subset N$, the **subgame of $\boldsymbol{v}$ relative to $\boldsymbol{N'}$**, denoted $v|_{N'}$, is defined by setting, for all $S \in 2^{N'}$, $v|_{N'}(S) \equiv v(S)$. A TU game v for N is **convex** if for all $i \in N$ and all $S, T \subseteq N \backslash \{i\}$, $S \subseteq T$ implies $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. Let $\boldsymbol{\mathcal{V}^N_{vex}}$ be the **class of convex games for $\boldsymbol{N}$**, and $\mathcal{V}_{vex} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{V}^N_{vex}$. A TU game v for N is **balanced** if for all non-negative function $\delta : 2^N \to \mathbb{R}_+$ such that for all $i \in N$, $\sum_{S \ni i} \delta(S) = 1$, $v(N) \geq \sum_{S \in 2^N} \delta(S) v(S)$.

Let $\mathbb{R}^N$ denote the Cartesian product of $|N|$ copies of $\mathbb{R}$, indexed by the members of N. For all $x \in \mathbb{R}^N$ and all $S \in 2^N$, let $x_S = (x_i)_{i \in S}$. A **payoff vector** for game v for N is an element x of $\mathbb{R}^N$ with $\sum_N x_i \leq v(N)$. A **solution,** denoted $\boldsymbol{\varphi}$, is a mapping defined on some domain of games that associates with each game in the domain a non-empty set of payoff vectors. A solution is **single-valued** if it associates with each game in its domain a unique payoff vector.

Given a game v for N, the **dual of $\boldsymbol{v}$**, denoted $\boldsymbol{v^d}$, is defined by setting, for all $S \subseteq N$,

$$v^d(S) \equiv v(N) - v(N \backslash S).$$

The number $v^d(S)$ is the amount that the complementary coalition $N \backslash S$ cannot prevent S from obtaining.

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^d \in \mathcal{V}$. Given a solution φ on $\mathcal{V}$, the **dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^d}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^d(v) \equiv \varphi(v^d).$$

A solution φ on $\mathcal{V}$ is **self-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^d(v)$.

An **axiom** is a desirable property of solutions. **Two axioms are dual** if whenever a solution satisfies one of them, the dual of this solution satisfies the other. **An axiom is self-dual** if it is its own dual.

[2] We use $\subseteq$ for weak set inclusion, and $\subset$ for strict set inclusion.

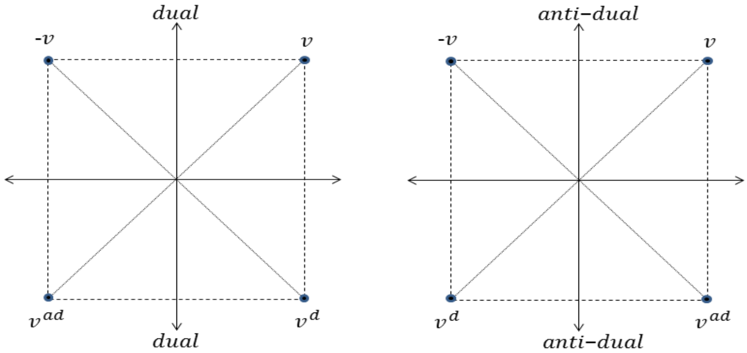

Fig. 1

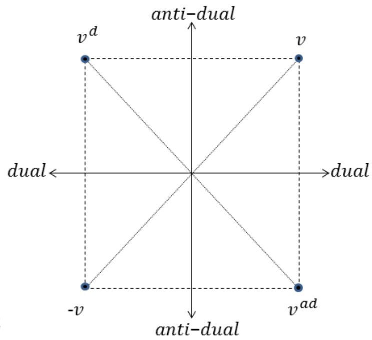

Fig. 2

Given a game v for N, the **anti-dual of v**, denoted $\boldsymbol{v^{ad}}$, is defined by setting, for all $S \subseteq N$,

$$v^{ad}(S) \equiv -v^d(S).$$

Let $\mathcal{V}$ be a class of games such that if $v \in \mathcal{V}$, then $v^{ad} \in \mathcal{V}$. The class of balanced games and the class of convex games satisfy this property. Given a solution φ on $\mathcal{V}$, the **anti-dual of φ**, denoted $\boldsymbol{\varphi^{ad}}$, is defined by setting, for all $v \in \mathcal{V}$,

$$\varphi^{ad}(v) \equiv -\varphi(v^{ad}).$$

A solution φ on $\mathcal{V}$ is **self-anti-dual** if for all $v \in \mathcal{V}$, $\varphi(v) = \varphi^{ad}(v)$.[3] **Two axioms are anti-dual** if whenever a solution satisfies one of them, the anti-dual of this solution satisfies the other. **An axiom is self-anti-dual** if it is its own anti-dual.

Figures 1 and 2 illustrate the relationship between v^d and v^{ad}. In Fig.1, the horizontal arrows show the opposite-sign relation. For instance, v^{ad} is v^d

[3]The definition of self-anti-dual solutions implies that φ is *efficient*, i.e. $\sum_N \varphi_i(v) = v(N)$.

with the opposite sign. In the left picture, the vertical arrow shows the duality relation. For instance, v^d is *dual* of v, and v is *dual* of v^d. Similarly, v^{ad} is *dual* of $-v$, and $-v$ is *dual* of v^{ad}. In the right picture, the vertical arrow shows the anti-duality relation. For instance, v^{ad} is *anti-dual* of v, and v is *anti-dual* of v^{ad}. Similarly, v^d is *anti-dual* of $-v$, and $-v$ is *anti-dual* of v^d. In Fig.2, the horizontal arrow shows the duality relation. The vertical arrow shows the anti-duality relation. Fig.2 summarizes the observation of Fig.1.

Finally, we introduce well-known solutions for TU games. The **core** (Gillies 1959) is defined as follows: for all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N$,

$$C(v) \equiv \Big\{ x \in \mathbb{R}^N \;\Big|\; \sum_{i \in N} x_i = v(N) \text{ and for all } S \subseteq N,\ \sum_{i \in S} x_i \geq v(S) \Big\}.$$

The core of v for N is not empty if and only if the game v is *balanced*.

The **Shapley value** (Shapley 1953) is defined as follows: for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $i \in N$,

$$Sh_i(v) \equiv \sum_{\substack{S \subseteq N \\ S \not\ni i}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \big[v(S \cup \{i\}) - v(S) \big].$$

On the domain of *convex* games, the core is never empty, and the Shapley value is a selection from the core.

Given $N \in \mathcal{N}$ and $v \in \mathcal{V}^N$, let $I(v)$ be the set of vectors $x \in \mathbb{R}^N$ such that for all $i \in N$, $x_i \geq v(\{i\})$, and $\sum_N x_i = v(N)$. For all $x \in I(v)$, let $e(v,x) \in \mathbb{R}^{2^N}$ be defined by setting, for all $S \subseteq N$, $e_S(v,x) \equiv v(S) - \sum_S x_i$. For all $z \in \mathbb{R}^{2^N}$, $\theta(z) \in \mathbb{R}^{2^N}$ is defined by rearranging the coordinates of z in non-increasing order. For all $z \in \mathbb{R}^{2^N}$, ***z* is lexicographically smaller than *z′*** if $\theta_1(z) < \theta_1(z')$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) < \theta_2(z')]$ or $[\theta_1(z) = \theta_1(z')$ and $\theta_2(z) = \theta_2(z')$ and $\theta_3(z) < \theta_3(z')]$, and so on. The **nucleolus** (Schmeidler 1969) is defined as follows:

$$Nu(v) \equiv \left\{ x \in I(v) \;\middle|\; \begin{array}{c} \text{For all } y \in I(v) \setminus \{x\},\ e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,y) \end{array} \right\}.$$

The nucleolus is a *single-valued* solution. On the domain of *convex* games, the nucleolus is a selection form the core.

3 Anti-dual of population monotonicity

In this section, we deal with a *single-valued* solution, denoted φ, on some domain of games. We write $x = \varphi(v)$ instead of $\{x\} = \varphi(v)$.

Population monotonicity says that for all game $v \in \mathcal{V}^N$ and all subgames

$v|_{N'} \in \mathcal{V}^{N'}$, if agents play in $v|_{N'}$, then the payoffs to the agents in v have to increase.

Population monotonicity: For all $N, N' \in \mathcal{N}$ such that $N' \subset N$, all $v \in \mathcal{V}^N$, and all $i \in N'$, $\varphi_i(v|_{N'}) \leq \varphi_i(v)$.

We introduce another monotonic property. We start with some game v for $N \in \mathcal{N}$. Next, we consider the game v^c played by $N' \subset N$. The worth of each coalition $S \subseteq N'$ is equal to the contribution of S to $N \backslash N'$ in v. This property says that in the game v^c, the payoffs to the members of N' have to be at least as large as in v.

Coalitional contribution monotonicity: For all $N, N' \in \mathcal{N}$ such that $N' \subset N$, all $v \in \mathcal{V}^N$, all $v^c \in \mathcal{V}^{N'}$ such that for all $S \subseteq N'$ $v^c(S) \equiv v(S \cup (N \backslash N')) - v(N \backslash N')$, and all $i \in N'$,

$$\varphi_i(v^c) \geq \varphi_i(v).$$

We clear that if v is *convex*, so is v^c.

Claim 1 *For all $N, N' \in \mathcal{N}$ such that $N' \subset N$, all $v \in \mathcal{V}^N_{vex}$, and all $S \subseteq N'$, $v^c(S) \equiv v(S \cup (N \backslash N')) - v(N \backslash N')$. Then, $v^c \in \mathcal{V}^{N'}_{vex}$.*

Proof of Claim 1. A game is *convex* if for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. For all $S, T \subseteq N'$,

$$\begin{aligned}
& v^c(S) + v^c(T) \\
= \; & v(S \cup (N \backslash N')) - v(N \backslash N') + v(T \cup (N \backslash N')) - v(N \backslash N') \\
\leq \; & v\left((S \cup (N \backslash N')) \cup (T \cup (N \backslash N'))\right) - v(N \backslash N') \\
& + v\left((S \cup (N \backslash N')) \cap (T \cup (N \backslash N'))\right) - v(N \backslash N') \\
= \; & v\left((S \cup T) \cup (N \backslash N')\right) - v(N \backslash N') \\
& + v\left((S \cap T) \cup (N \backslash N')\right) - v(N \backslash N') \\
= \; & v^c(S \cup T) + v^c(S \cap T),
\end{aligned}$$

the desired conclusion. ■

Using the anti-duality operator, we obtain the following result:

Proposition 1 *On the domain of convex games, population monotonicity and coalitional contribution monotonicity are anti-dual properties.*

Proof. Let φ be a *population monotonic* solution on $\mathcal{V}^N_{vex}$. Let $N' \subset N$. Let $v \in \mathcal{V}^N_{vex}$, $x \equiv \varphi^{ad}_{N'}(v)$ and $y \equiv \varphi^{ad}(v|_{N'})$.[4] For all $S \subseteq N'$, $w(S) \equiv v^{ad}|_{N'}(S)$. By the definition of φ^{ad}, $-x \equiv \varphi_{N'}(v^{ad})$ and $-y \equiv \varphi(w)$. Note that $v^{ad} \in \mathcal{V}^N_{vex}$ and $w \in \mathcal{V}^{N'}_{vex}$. Since φ is *population monotonic*, for all $i \in N'$, $\varphi_i(w) \leq \varphi_i(v^{ad})$. By the definition of φ^{ad}, for all $i \in N'$, $\varphi^{ad}_i(w^{ad}) \geq \varphi^{ad}_i(v)$.

For all $S \subseteq N'$,

$$\begin{aligned} w^{ad}(S) &= -w(N') + w(N' \backslash S) \\ &= v^d|_{N'}(N') - v^d|_{N'}(N' \backslash S) \\ &= v(N) - v(N \backslash N') - v(N) + v(N \backslash (N' \backslash S)) \\ &= v(S \cup (N \backslash N')) - v(N \backslash N'), \end{aligned}$$

the desired conclusion. ■

The following properties of solutions are well known (e.g., Peleg and Sudhölter 2003).

Efficiency: For all $N \in \mathcal{N}$, and all $v \in \mathcal{V}^N$, $\sum_N \varphi_i(v) = v(N)$.

Individual rationality: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $i \in N$, $\varphi_i(v) \geq v(\{i\})$.

Reasonableness: For all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, and all $i \in N$, $\varphi_i(v) \leq v(N) - v(N \backslash \{i\})$.[5]

Lemma 1 *On the domain of convex games, (i) efficiency is self-anti-dual, and (ii) individual rationality and reasonableness are anti-dual properties.*

Proof. Immediately from the definition of anti-duality. ■

For all $N, N' \in \mathcal{N}$ such that $N' \subset N$, and all $v \in \mathcal{V}^N$, the **self-reduced game of $\boldsymbol{v}$ relative to $\boldsymbol{\varphi}$ and $\mathbf{N'}$**, denoted $r^{\varphi}_{N'}(v)$, is defined by setting, for all $S \subseteq N'$,

$$r^{\varphi}_{N'}(v)(S) \equiv \begin{cases} v(N) - \sum_{N \backslash N'} \varphi_i(v) & \text{if } S = N' \\ v\left(S \cup (N \backslash N')\right) - \sum_{N \backslash N'} \varphi_i(v\left|_{S \cup (N \backslash N')}\right.) & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

[4]Note that $\varphi^{ad}_{N'}(v) \equiv \left(\varphi^{ad}_i(v)\right)_{i \in N'}$.

[5]On the domain of convex games, it is "reasonableness from above" (Milnor 1952).

We require that the outcome a solution chooses for each game in $\mathcal{V}^N$ should be equal to the outcome chosen by the solution for the *self-reduced game relative to* φ *and* N'.[6]

Self consistency: For all $N, N' \in \mathcal{N}$ such that $N' \subset N$, and all $v \in \mathcal{V}^N$, we have $r^{\varphi}_{N'}(v) \in \mathcal{V}^{N'}$ and for all $i \in N'$, $\varphi_i(r^{\varphi}_{N'}(v)) = \varphi_i(v)$.

On the domain of all TU games, the Shapley value is *self consistent*. However, on the domain of *convex* games, it is not, since $\mathcal{V}_{vex}$ is not closed under the *self-reduction operator* for this solution.

The following notion is a weaker notion:

Bilateral self-consistency: For all $N, N' \in \mathcal{N}$ such that $N' \subset N$ with $|N'| = 2$, and all $v \in \mathcal{V}^N$, we have $r^{\varphi}_{N'}(v) \in \mathcal{V}^{N'}$ and for all $i \in N'$, $\varphi_i(r^{\varphi}_{N'}(v)) = \varphi_i(v)$.

On the domain of *convex* games, the Shapley value is *bilaterally self-consistent.*

Let us consider the following alternative notion of consistency.[7]

Transfer agreement consistency (Oishi et al. 2013)**:** For all $N, N' \in \mathcal{N}$ such that $N' \subset N$, and all $v \in \mathcal{V}^N$, if for all $S \subseteq N'$,

$$\tilde{r}^{\varphi}_{N'}(v)(S) \equiv \begin{cases} v(N) - \sum_{N\setminus N'} \varphi_i(v) & \text{if } S = N', \\ v(S) + \sum_{N\setminus N'} \varphi_i(v^{N\setminus S}) - \sum_{N\setminus N'} \varphi_i(v) & \text{if } S \neq N', \emptyset, \\ 0 & \text{if } S = \emptyset, \end{cases}$$

where $v^{N\setminus S}$ is the game for $N\setminus S$ defined by setting, for all $T \subseteq N\setminus S$, $v^{N\setminus S}(T) \equiv v(S \cup T) - v(S)$, we have $\tilde{r}^{\varphi}_{N'}(v) \in \mathcal{V}^{N'}$ and for all $i \in N'$, $\varphi_i(\tilde{r}^{\varphi}_{N'}(v)) = \varphi_i(v)$.

We require the notion that weakens *transfer agreement consistency* by limiting its application to subpopulation of two agents. The scenario underlying the reduced game $\tilde{r}^{\varphi}_{N'}$ is as follows.[8] Imagine that agent i announces that he will cooperate with anybody if he obtains $v(\{i\})$. If some agents, who form a coalition $T \subseteq N\setminus\{i\}$, cooperate with agent i, the coalition $\{i\} \cup T$ obtains $v(\{i\} \cup T)$. Since the reward of agent i for his cooperation is $v(\{i\})$, the coalition T obtains the remainder $v(\{i\} \cup T) - v(\{i\})$. Thus, each agent $j \in N\setminus\{i\}$ plays $v^{N\setminus\{i\}}$ and obtains $\varphi_j(v^{N\setminus\{i\}})$. If agent i does not make this announcement, each agent $j \in N\setminus\{i\}$ obtains $\varphi_j(v)$. If agents i and $k \in N\setminus N'$ agree that the difference $\varphi_k(v^{N\setminus\{i\}}) - \varphi_k(v)$ should be transferred from agent k to agent

[6] The self-reduced game and self-reduced consistency are usually called "HM-reduced game" and "HM-consistency", respectively (Hart and Mas-Colell 1989). We use the terminology introduced by Thomson (1996).

[7] On the domain of all TU games, *self-consistency* and *transfer-agreement consistency* are *dual* properties. For details, see Oishi et al. (2013).

[8] The scenario here is introduced by Oishi et al. (2013).

i, then agent i obtains $\tilde{r}^{\varphi}_{N'}(v)(\{i\})$ as defined above. The worth $\tilde{r}^{\varphi}_{N'}(v)(\{j\})$ can be interpreted in the same manner. *Bilateral transfer-agreement consistency* requires that what agents i and j get should be unchanged if such an agreement between agents i and $k \in N\backslash N'$ or between agents j and $k \in N\backslash N'$ takes place.

Bilateral transfer-agreement consistency: For all $N, N' \in \mathcal{N}$ such that $N' \subset N$ with $|N'| = 2$, and all $v \in \mathcal{V}^N$, we have $\tilde{r}^{\varphi}_{N'}(v) \in \mathcal{V}^{N'}$ and for all $i \in N'$ $\varphi_i(\tilde{r}^{\varphi}_{N'}(v)) = \varphi_i(v)$.

Proposition 2 *On the domain of convex games, bilateral self-consistency and bilateral transfer-agreement consistency are anti-dual properties.*

Proof. See Appendix. ■

Hokari and Gellekom (2002) identified the following sufficient conditions for a solution to be *population monotonic* on the domain of *convex* games.

Proposition A *(Hokari and Gellekom 2002) On the domain of convex games, if a solution is efficient, individual rational, and bilaterally self-consistent, then it is population monotonic.*

Corollary 1 *On the domain of convex games, the Shapley value is population monotonic (Sprumont 1990).*

Using the *anti-duality* operator, we derive sufficient conditions for a solution to be *coalitional contribution monotonic* on the domain of *convex* games from Hokari and Gellekom's conditions.

Proposition 3 *(Anti-dual of Proposition A) On the domain of convex games, if a single-valued solution is efficient, reasonable, and bilaterally transfer-agreement consistent, then it is coalitional contribution monotonic.*

Proof. *By Propositions 1 and 2, and Lemma 1.* ■

On the domain of *convex* games $\mathcal{V}_{vex}$, the Shapley value is *self-anti-dual* (Oishi and Nakayama 2009, Theorem 2). On $\mathcal{V}_{vex}$, it is *efficient*, *reasonable*, and *bilaterally transfer-agreement consistent*. Thus, the conditions stated in Proposition 3 are satisfied by the Shapley value.

4 Duality and anti-duality approach to allocation problems

In this section, we take the duality and anti-duality approach to an analysis of rules for allocation problems. First, we axiomatize the Shapley rule for several allocation problems using the notion of *duality*.[9] Next, we analyze the relationship between the nucleolus rules for claims problems and for public good problems using the notion of *anti-duality*.

4.1 Duality, and anti-duality for allocation problems

We introduce the notions of *duality* and *anti-duality* for solutions and axioms for allocation problems. An **allocation problem for N** is a pair (N, p), where $N \in \mathcal{N}$ is a finite non-empty set of agents and $p = (p_i)_{i \in N}$ is a profile of parameters for N. For each $i \in N$, the parameter p_i is the benefit or the cost experienced by agent $i \in N$ when engaging in some economic activity. Let $\mathcal{P}$ be the set of all allocation problems on $\mathcal{N}$.

Given all $S \in 2^N$, we denote by $v_P : \mathcal{P} \to \mathbb{R}^{2^N}$ a mapping that associates with each allocation problem (N, p) in the domain the unique $2^{|N|}$-dimensional vector whose S-component is the amount coalition S can obtain on its own. By convention, $v_P(N, p)(\emptyset) = 0$. The number $v_P(N, p)$ is the **coalitional game for N derived from the allocation problem (N, p)**.

Let $\boldsymbol{\mathcal{V}_P}$ be the class of all coalitional games derived from allocation problems $\mathcal{P}$. Given $(N, p) \in \mathcal{P}$, an **allocation** for (N, p) is a vector $x \in \mathbb{R}^N$ such that $\sum_N x_i = v_P(N, p)(N)$. Let $\mathbf{X(N, p)}$ be the set of allocations for (N, p). A **solution for coalitional games** is a mapping $\phi : \mathcal{V}_P \to \mathbb{R}^N$ that associates with each coalitional game $v_P(N, p)$ in the domain a unique allocation in $X(N, p)$. We refer to the composite mapping $\varphi \equiv \phi \circ v_P$ as an **allocation rule, or simply a rule, for allocation problems on the domain of $\boldsymbol{\mathcal{P}}$**. For instance, we refer to the composite mapping $\varphi \equiv Sh \circ v_P$ as the **Shapley rule**, and to the composite mapping $\varphi \equiv Nu \circ v_P$ as the **nucleolus rule**.

Given a rule φ on $\mathcal{P}$, **the dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^d}$, is defined by setting, for all $(N, p) \in \mathcal{P}$,

$$\varphi^d(N, p) \equiv \phi[(v_P^d)(N, p)].$$

A rule φ on $\mathcal{P}$ is **self-dual** if for all $(N, p) \in \mathcal{P}$, $\varphi(N, p) = \varphi^d(N, p)$. **Two axioms are dual** if whenever a rule satisfies one of them, the dual of this rule satisfies the other. **An axiom is self-dual** if it is its own dual.

Given a rule φ on $\mathcal{P}$, **the anti-dual of $\boldsymbol{\varphi}$**, denoted $\boldsymbol{\varphi^{ad}}$, is defined by

[9] One can axiomatize the Shapley value for these problems by using the notion of *anti-duality.* For simplicity of our analysis, we take the *duality* approach here.

setting, for all $(N, p) \in \mathcal{P}$,

$$\varphi^{ad}(N, p) \equiv -\phi[(v_P^{ad})(N, p)].$$

A rule φ on $\mathcal{P}$ is **self-anti-dual** if for all $(N, p) \in \mathcal{P}$, $\varphi(N, p) = \varphi^{ad}(N, p)$. **Two axioms are anti-dual** if whenever a rule satisfies one of them, the anti-dual of this rule satisfies the other. **An axiom is self-anti-dual** if it is its own anti-dual.

4.2 Allocation problems

We consider the following classes of allocation problems.

4.2.1 Airport problems

There is a set of airlines for whom an airstrip they will jointly use is to be built. Each airline owns one type of aircraft. Airlines have different needs for airstrips, since they own different types of aircraft. An airstrip needed to accommodate the largest aircraft is to be built. The problem is to determine how to share the cost of the airstrip between the airlines (Littlechild and Owen 1973).

An **airport problem** is a pair (N, c), where $N \in \mathcal{N}$ is the set of airlines and $c = (c_i)_{i \in N}$ is the profile of cost parameters, namely c_i is the construction cost of the airstrip for airline i. We assume that the cost is increasing in the length of the airstrip. For simplicity, we require that for each $i \in N$, $c_{i+1} < c_i$ with $c_{n+1} \equiv 0$. Let $\mathcal{C}$ be the class of all airport problems on $\mathcal{N}$.

Given $(N, c) \in \mathcal{C}$, the **airport game** is defined by setting, for all $S \subseteq N$,

$$c_A(N, c)(S) \equiv \max_{i \in S} c_i.$$

For all $S \in 2^N$, $c_A(N, c)(S)$ represents the cost of the airstrip needed to accommodate the members of coalition S. It is equal to the cost of the airstrip needed to accommodate the member of the coalition whose cost parameter is the largest.

Let $\mathcal{C}_A$ be the class of all airport games. Given $(N, c) \in \mathcal{C}$, an allocation for (N, c) is a vector $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = \max_N c_i$ (which is equal to c_1). Let $X(N, c)$ be the set of allocations for (N, c). A **solution for airport games** is a mapping $\phi_A : \mathcal{C}_A \to \mathbb{R}^N$ that associates with each airport game $c_A(N, c)$ in the domain an allocation in $X(N, c)$. We refer to the composite mapping $\varphi_A \equiv \phi_A \circ c_A$ as a **rule for airport problems**. The **Shapley rule for airport problems** is defined by $\varphi_A^{Sh} \equiv Sh \circ c_A$.

4.2.2 Bidding ring problems

An **English auction** is an oral auction in which an auctioneer initially sets a bid at a seller's reservation price and then gradually increases the price until only one bidder remains active. There is a set of buyers in a *single-object English auction*. There is no asymmetry of information between the buyers; that is, each buyer has information on the valuations of all buyers for the object. The valuation of each buyer is positive, and all valuations are different. The reservation price is zero. A bidding ring is formed by all buyers. The bidding ring wins the auction by making the buyer whose valuation is the highest the sole bidder. The benefit of the ring members' strategy is equal to the valuation of this buyer. The problem for the members in the ring is to determine how to share the benefit of their strategy (Graham et al. 1990).

A **bidding ring problem** is a pair (N, c), where $N \in \mathcal{N}$ is the set of buyers and $c = (c_i)_{i \in N}$ is the profile of valuations for a single object, c_i being the valuation of buyer i. For simplicity, we require that for each $i \in N$, $c_{i+1} < c_i$ with $c_{n+1} \equiv 0$. Let $\mathcal{C}$ be the class of all bidding ring problems on $\mathcal{N}$.

Given $(N, c) \in \mathcal{C}$, the **bidding ring game** is defined by setting, for all $S \subseteq N$,

$$v_B(N, c)(S) = \begin{cases} c_1 - \max_{j \notin S} c_j & \text{if } S \ni 1 \\ 0 & \text{if } S \not\ni 1, \end{cases}$$

where $\max_{j \notin N} c_j \equiv 0$. The intuition is as follows: First, under the English auction rule, it is a dominant strategy for each bidder to remain active until bidding reaches his valuation. Second, any coalition including buyer 1 can win the auction, and achieve the net benefit $c_1 - \max_{j \notin S} c_j$ by making buyer 1 the sole bidder in the coalition and his bidding c_1. Finally, no coalition that does not include buyer 1 wins the auction, and hence its net benefit is 0.

Let $\mathcal{V}_B$ be the class of all bidding ring games. Given $(N, c) \in \mathcal{C}$, an allocation for (N, c) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = c_1$. Let $X(N, c)$ be the set of allocations for (N, c) . A **solution for bidding ring games** is a mapping $\phi_B : \mathcal{V}_B \to \mathbb{R}^N$ that associates with each bidding ring game $v_B(N, c)$ in the domain an allocation in $X(N, c)$. We refer to the composite mapping $\varphi_B \equiv \phi_B \circ v_B$ as a **rule for bidding ring problems**. The **Shapley rule for bidding ring problems** is defined by $\varphi_B^{Sh} \equiv Sh \circ v_B$.

4.2.3 Liability problems

Someone suffers a cumulative injury that is caused by several persons in succession.[10] Each injuring party has taken a wrongful act causing some damage, and the sequence of wrongful acts has resulted in the injury. The injured

[10] We deal with the simplest case of liability problems introduced by Dehez and Ferey (2013).

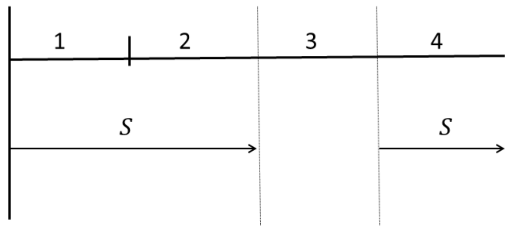

*Fig.*3

party is entitled to compensation. The problem is to determine how to share the compensation between the injuring parties (Dehez and Ferey 2013).

A **liability problem** is a pair (N, d), where $N \in \mathcal{N}$ is the set of injuring parties and $d = (d_i)_{i \in N}$ is the profile of damage parameters such that for each $i \in N$, $d_i > 0$. Let $\mathcal{D}$ be the class of all liability problems on $\mathcal{N}$.

The detail of the scenario underlying a liability problem is as follows: Someone suffers an injury. Agent 1 has taken a wrongful act that is the root of the injury. Let d_1 be the damage that agent 1 has caused. After agent 1's wrongful act, agent 2 has taken a wrongful act. Without agent 1's wrongful act, agent 2's wrongful act would not have occurred. Let d_2 be the additional damage that agent 2 has caused. Agents 1 and 2 have caused the cumulative damage $d_1 + d_2$. After agents 1 and 2's wrongful acts, agent 3 has taken a wrongful act. Without agents 1 and 2's wrongful acts, agent 3's wrongful act would not have occurred. Let d_3 be the additional damage that agent 3 has caused. Agents 1, 2, and 3 have caused the cumulative damage $d_1 + d_2 + d_3$. The process continues until agent n. The agents in N have caused the cumulative damage $d_1 + d_2 + \cdots + d_n$.

Given $(N, d) \in \mathcal{D}$, the **liability game** is defined by setting, for all $S \subseteq N$,

$$v_L(N, d)(S) \equiv \begin{cases} \sum_N d_k & \text{if } S = N, \\ \sum_{k=1}^{(\min N \setminus S)-1} d_k & \text{if } S \ni 1 \text{ and } S \subset N, \\ 0 & \text{if } S \not\ni 1, \end{cases}$$

where $\min S$ is the smallest number of S. For all $S \in 2^N$, $v_L(N, d)(S)$ represents the cumulative damage that the agents in S have caused.

Consider a four-agent example (Fig.3). Let $N = \{1, 2, 3, 4\}$, and $S = \{1, 2, 4\}$. Agent 4's wrongful act can only occur if agents 1, 2, and 3 have committed wrongful acts. In this case, however, agent 4 does not cause any damage. This is because agent 3 is not included in S. Thus, the cumulative damage that the agents in S have caused is $d_1 + d_2$. In general, we denote by $(\min N \setminus S) - 1$ the last agent who appears in the sequence of wrongful acts that the agents in S have caused. For instance, the last agent in Fig.3 is agent $(\min\{1,2,3,4\}\backslash\{1,2,4\}) - 1$, namely agent 2.

Let $\mathcal{V}_L$ be the class of all liability games. Given $(N,d) \in \mathcal{D}$, an allocation for (N,d) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \sum_N d_i$. Let $X(N,d)$ be the set of allocations for (N,d). A **solution for liability games** is a mapping $\phi_L : \mathcal{V}_L \to \mathbb{R}^N$ that associates with each liability game $v_L(N,d)$ in the domain an allocation in $X(N,d)$. We refer to the composite mapping $\varphi_L \equiv \phi_L \circ v_L$ as a **rule for liability problems**. The **Shapley rule for liability problems** is defined by $\varphi_L^{Sh} \equiv Sh \circ v_L$.

4.2.4 Polluted river problems

Imagine a line divided into several segments. Each segment (or watercourse) belongs to each state. The water flows from the most upstream state to the most downstream state. Each state produces some amount of pollutants. Each state is responsible for cleaning not only its own watercourse but also all downstream watercourses. The problem is to determine how to share the total cleaning cost between the states (Ni and Wang 2007).

A **polluted river problem** is a pair (N,d), where $N \in \mathcal{N}$ is the set of states and $d = (d_i)_{i\in N}$ is the profile of pollutants parameters such that for each $i \in N$, $d_i > 0$. Let $\mathcal{D}$ be the class of all polluted river problems on $\mathcal{N}$.

Given $(N,d) \in \mathcal{D}$, the **polluted river game (simply, the river game)** is defined by setting, for all $S \subseteq N$,

$$c_R(N,d)(S) = \sum_{\min S}^{n} d_i.$$

For all $S \in 2^N$, $c_R(N,d)(S)$ represents the total cleaning cost that all downstream states of S need.[11]

Let $\mathcal{C}_R$ be the class of all river games. Given $(N,d) \in \mathcal{D}$, an allocation for (N,d) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = \sum_N d_i$. Let $X(N,d)$ be the set of allocations for (N,d). A **solution for river games** is a mapping $\phi_R : \mathcal{C}_R \to \mathbb{R}^N$ that associates with each river game $c_R(N,d)$ in the domain an allocation in $X(N,d)$. We refer to the composite mapping $\varphi_R \equiv \phi_R \circ c_R$ as a **rule for polluted river problems**. The **Shapley rule for polluted river problems** is defined by $\varphi_R^{Sh} \equiv Sh \circ c_R$.

4.2.5 A three-dimensional box for coalitional games and rules

Next, we observe the *duality* between TU games derived from the allocation problems mentioned above.

[11] This game is derived from the notion of "downstream responsibility" (Ni and Wang 2007).

Remark 1 *The following assertions hold:*

(i) The class $\mathcal{C}_A$ of airport games and the class $\mathcal{V}_B$ of bidding ring games are dual.
(ii) The class $\mathcal{V}_L$ of liability games and the class $\mathcal{C}_R$ of river games are dual.
(iii) The Shapley value of airport games on the domain $\mathcal{C}_A$ coincides with that of bidding ring games on the domain $\mathcal{V}_B$.
(iv) The Shapley value of liability games on the domain $\mathcal{V}_L$ coincides with that of river games on the domain $\mathcal{C}_R$.

The proof of (i) and (ii) follows from simple calculation. The proof of (iii) and (iv) follows from the *self-duality* of the Shapley value.

In the following remark, we uncover the hidden structure of the Shapley rules for allocation problems mentioned above.

Remark 2 *The following assertions hold:*

(i) There exists a bijection $f : \mathcal{D} \to \mathcal{C}$ such that for all $N \in \mathcal{N}$ and all $i \in N$, $c_i = d_i + d_{i+1} + \cdots + d_n$.

Using the bijection f defined in (i), the following assertions hold:

(ii) For all $(N, d) \in \mathcal{D}$, $c_A(f(N, d)) = c_R(N, d)$, and for all $(N, c) \in \mathcal{C}$, $c_R(f^{-1}(N, c)) = c_A(N, c)$.
(iii) For all $(N, d) \in \mathcal{D}$, $v_B(f(N, d)) = v_L(N, d)$, and for all $(N, c) \in \mathcal{C}$, $v_L(f^{-1}(N, c)) = v_B(N, c)$.
(iv) For all $(N, d) \in \mathcal{D}$, $\varphi^{Sh}_A(f(N, d)) = \varphi^{Sh}_B(f(N, d)) = \varphi^{Sh}_L(N, d) = \varphi^{Sh}_R(N, d)$.
(v) For all $(N, c) \in \mathcal{C}$, $\varphi^{Sh}_A(N, c) = \varphi^{Sh}_B(N, c) = \varphi^{Sh}_L(f^{-1}(N, c)) = \varphi^{Sh}_R(f^{-1}(N, c))$.

Using the bijection f defined in (i), we can associate to a river game on the domain $\mathcal{C}_R$ an airport game on the domain $\mathcal{C}_A$ as follows[12]: For all $N \in \mathcal{N}$, all $(N, d) \in \mathcal{D}$, and all $i \in N$, let $c_i = f_i(N, d) = d_i + d_{i+1} + \cdots + d_n$. An airport game $c_A \in \mathcal{C}_A$ is rewritten by setting, for all $S \subseteq N$, $c_A(f(N, d))(S) = \max_S[d_i + d_{i+1} + \cdots + d_n] = \sum^n_{\min S} d_i$, which is a river game $c_R(N, d)$. We can also associate to an airport game on the domain $\mathcal{C}_A$ a river game on the domain $\mathcal{C}_R$ as follows: For all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, and all $i \in N$, let $d_i = f^{-1}_i(N, c) = c_i - c_{i+1}$ with $c_{n+1} \equiv 0$. A river game $c_R \in \mathcal{C}_R$ is rewritten by setting, for all $S \subseteq N$, $c_R(f^{-1}(N, c))(S) = \sum^n_{\min S}[c_i - c_{i+1}] = \max_S c_i$, which is an airport game $c_A(N, c)$. Similarly, using the bijection f, we can associate to a liability game on the domain $\mathcal{V}_L$ a bidding ring game on the domain $\mathcal{V}_B$, and we can associate to a bidding ring game on the domain $\mathcal{V}_B$ a liability game on the domain $\mathcal{V}_L$. The assertions (iv) and (v) in Remark 2 follow from Remark 1 (iii)-(iv) and Remark 2 (ii)-(iii).

[12] This observation was pointed out by van den Brink and van der Laan (2008).

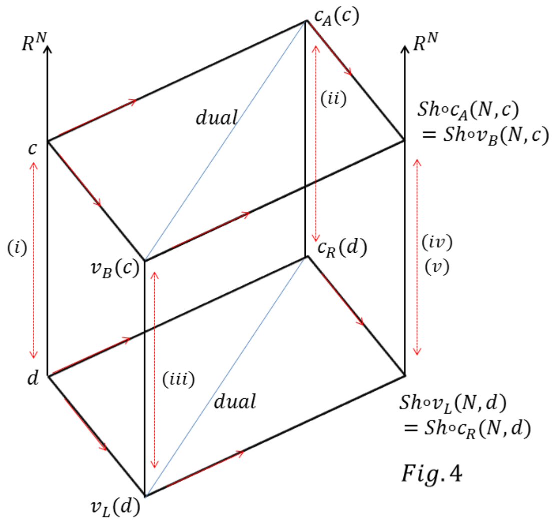

Fig.4

In Fig.4, the three-dimensional box illustrates the *duality* relation between the classes of allocation problems mentioned above. It summarizes the observations in Remarks 1 and 2.

4.3 Duality approach to bidding ring problems

In the literature, the Shapley rule for bidding ring problems has not been axiomatized. Just by identifying the *dual* of each axiom involved in an axiomatization of φ_A^{Sh}, we obtain an axiomatization of φ_B^{Sh}.

Let us consider the *dual* of each axiom involved in the axiomatization of the Shapley rule for airport problems (Chun et al. 2012).

First, we consider the following property. Each airline i has the right to use at least the airstrip to accommodate the airline i. It says that each airline i should pay at least an equal share of c_i.

Equal share lower bound for airport problems: For all $(N, c) \in \mathcal{C}$ and all $i \in N$,

$$\varphi_{A[i]}(N, c) \geq \frac{c_i}{n}.$$

The *dual* of *the equal share lower bound* says that each buyer $i \in N$ should gain at least an equal share of his valuation between all buyers. It is *self-dual*.

Equal share lower bound for bidding ring problems: For all $(N, c) \in \mathcal{C}$ and all $i \in N$,

$$\varphi_{B[i]}(N, c) \geq \frac{c_i}{n}.$$

Next, we consider the following property for airport problems. It requires that if the cost of an airline increases, then all the other airlines should pay at most as much as they did initially.

Individual monotonicity for airport problems: Fix an arbitrary $N \in \mathcal{N}$. For all $(N, c) \in \mathcal{C}$, all $(N, c') \in \mathcal{C}$, and all $i \in N$, if $c'_i > c_i$, and for all $j \in N \backslash \{i\}$, $c'_j = c_j$, then for all $j \in N \backslash \{i\}$,

$$\varphi_{A[j]}(N, c') \leq \varphi_{A[j]}(N, c).$$

The *dual* of *individual monotonicity* says that if the valuation of a buyer increases, then all the other buyers should share at most as much as they did initially. It is *self-dual.*

Individual monotonicity for bidding ring problems: Fix an arbitrary $N \in \mathcal{N}$. For all $(N, c) \in \mathcal{C}$, all $(N, c') \in \mathcal{C}$, and all $i \in N$, if $c'_i > c_i$, and for all $j \in N \backslash \{i\}$, $c'_j = c_j$, then for all $j \in N \backslash \{i\}$,

$$\varphi_{B[j]}(N, c') \leq \varphi_{B[j]}(N, c).$$

Our final property for airport problems says that if a new airline arrives, then all airlines whose costs are more than the cost of the new airline should be affected equally.

Population fairness for airport problems: For all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{c_j, c_k\} > c_i$,

$$\varphi_{A[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{A[j]}(N, c) = \varphi_{A[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{A[k]}(N, c).$$

The *dual* of *population fairness* says that if a new buyer arrives, then all buyers whose evaluations are more than the valuation of the new buyer should be affected equally. It is *self-dual.*

Population fairness for bidding ring problems: For all $N \in \mathcal{N}$, all $(N, c) \in \mathcal{C}$, all $i \in \mathcal{I} \backslash N$, all $j, k \in N$ such that $\min\{c_j, c_k\} > c_i$,

$$\varphi_{B[j]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{B[j]}(N, c) = \varphi_{B[k]}(N \cup \{i\}, c_{N \cup \{i\}}) - \varphi_{B[k]}(N, c).$$

Thus, we obtain the following axiomatization of solution φ_B^{Sh} that is *dual* of the axiomatization of solution φ_A^{Sh}.

Theorem A *(Chun et al. 2012) For airport problems, the Shapley rule is the only rule satisfying the equal share lower bound, individual monotonicity, and population fairness.*

Theorem 1 *(Self-dual of Theorem A) For bidding ring problems, the Shapley rule is the only rule satisfying the equal share lower bound, individual monotonicity, and population fairness.*

4.4 Duality approach to liability problems and polluted river problems

In the literature, the Shapley rule for liability problems has not been axiomatized. We will derive an axiomatization of it.

Our strategy is simple. Fragnelli and Marina (2010) characterized the Shapley rule for airport problems by three properties: *the equal share lower bound, the equal share upper bound,* and *last-airline consistency.*[13] For all $N \in \mathcal{N}$, each property depends only on the profile of parameters c. By Remark 2-(iv), we have that for all $(N, d) \in \mathcal{D}$, $Sh \circ v_L(N, d) = Sh \circ c_A(f(N, d))$. Using Fragnelli and Marina's characterization, we derive axioms that allow an axiomatization of $Sh \circ v_L(N, d)$.

Our first property is as follows: The contribution of agent i to the cumulative injury is the difference between the entire damage (i.e. $d_1 + d_2 + \cdots + d_n$) and the cumulative damage that the agents except for agent i have caused (i.e. $d_1 + d_2 + \cdots + d_{i-1}$). Thus, we refer to $d_i + d_{i+1} + \cdots + d_n$ as *agent i's contribution* to the cumulative injury. The following property requires that each agent $i \in N$ should pay at least an equal share of his contribution to the cumulative injury.

Equal share lower bound for liability problems: For all $(N, d) \in \mathcal{D}$ and all $i \in N$,

$$\varphi_{L[i]}(N, d) \geq \frac{d_i + d_{i+1} + \cdots + d_n}{n}.$$

Next, we consider the following property for airport problems. Each airline $i \in N$ needs an airstrip of cost at least c_i. On the other hand, airlines $i+1, i+2, \cdots, n$ could free ride on airline i since the cost of the airstrip to accommodate them is less than c_i. The following property requires that airline i should pay at most an equal share of c_i between the airlines without free riders.

Equal share upper bound for airport problems: For all $(N, c) \in \mathcal{C}$ and all $i \in N$,

$$\varphi_{A[i]}(N, c) \leq \frac{c_i}{i}.$$

[13] In our model, the "last-airline" is airline n since $c_1 > c_2 \cdots > c_n$. In the literature (for instance, Thomson 2007), the airline that needs the shortest airstrip is often referred to as airline 1 (the "first-airline") under the assumption that $c_n > c_{n-1} \cdots > c_1$.

Our next property for liability problems is as follows: The sequential wrongful acts from agent 1 to agent $i-1$ has triggered agent i's wrongful act. On the other hand, agents $i+1, i+2, \cdots, n$'s wrongful acts are not the cause of agent i's wrongful act. The following property requires that agent i should pay at most an equal share of his contribution to the cumulative injury between agents $1, 2, \cdots, i$.

Equal share upper bound for liability problems: For all $(N, d) \in \mathcal{D}$ and all $i \in N$,

$$\varphi_{L[i]}(N, d) \leq \frac{d_i + d_{i+1} + \cdots + d_n}{i}.$$

Finally, we consider the following property for airport problems. Imagine that airline n pays the cost $\varphi_{A[n]}(N, c)$ and leaves. Furthermore, imagine that $\varphi_{A[n]}(N, c)$ is used to cover the cost of construction for the segment airline n uses. Airline n's contribution to the segment it uses implies contributing to the segments each airline $i \in N \backslash \{n\}$ uses. Thus, the cost parameter of airline $i \in N \backslash \{n\}$ is adjusted to $c'_i = c_i - \varphi_{A[n]}(N, c)$. It says that the outcome the rule chooses for each problem should be invariant under the departure of the last airline.

Last-airline consistency: For all $(N, c) \in \mathcal{C}$ with $n \geq 2$, and all $i \in N'$,

$$\varphi_{A[i]}(N, c) = \varphi_{A[i]}(N', c'),$$

where $N' = N \backslash \{n\}$, $c'_i = c_i - \varphi_{A[n]}(N, c)$ for all $i \in N'$, and $(N', c') \in \mathcal{C}$.

Our final property for liability problems is as follows: Imagine that agent n pays the compensation $\varphi_{L[n]}(N, d)$ and leaves. Furthermore, imagine that the remaining damage $d_n - \varphi_{L[n]}(N, d)$ is added to the additional damage caused by agent $n-1$. As a result, the additional damage caused by agent $n-1$ is adjusted to $d_{n-1} + d_n - \varphi_{L[n]}(N, d)$. The following property requires that the outcome the rule chooses for each problem should be invariant under the departure of the last-injuring party.

Last-injuring party consistency: For all $(N, d) \in \mathcal{D}$ with $n \geq 2$, and all $i \in N'$,

$$\varphi_{L[i]}(N, d) = \varphi_{L[i]}(N', d'),$$

where $N' = N \backslash \{n\}$ and $d' = (d_1, d_2, \cdots, d_{n-2}, d_{n-1} + d_n - \varphi_{L[n]}(N, d)) \in \mathcal{D}$.

Thus, we obtain the following axiomatization of solution φ_L^{Sh}.

Theorem B *(Fragnelli and Marina 2010) For airport problems, the Shapley rule is the only rule satisfying the equal share lower bound, the equal share upper bound, and last-airline consistency.*

Theorem 2 *(Related dual of Theorem B) For liability problems, the Shapley rule is the only rule satisfying the equal share lower bound, the equal share upper bound, and last-injuring party consistency.*[14]

By identifying the *dual* of each axiom in Theorem 2, we obtain a new axiomatization of the Shapley rule for polluted river problems.

Our first property is as follows: Since each state $i \in N$ is responsible for not only its own cleaning cost but also all downstream costs, we refer to $d_i + d_{i+1} + \cdots + d_n$ as state i's responsibility. The following property requires that each state i should pay at least an equal share of his responsibility. It is *self-dual.*

Equal share lower bound for polluted river problems: For all $(N, d) \in \mathcal{D}$ and all $i \in N$,

$$\varphi_{R[i]}(N, d) \geq \frac{d_i + d_{i+1} + \cdots + d_n}{n}.$$

Our next property for polluted river problems is as follows: Since the upstream states of state i, states $1, 2, \cdots, i-1$, are responsible for all downstream costs, they share the responsibility for $d_i + d_{i+1} + \cdots + d_n$. On the other hand, the downstream states of state i, states $i+1, i+2, \cdots, n$, are not responsible for $d_i + d_{i+1} + \cdots + d_n$. The following property requires that state i should pay at most an equal share of his responsibility for cleaning between states $1, 2, \cdots, i$. It is *self-dual.*

Equal share upper bound for polluted river problems: For all $(N, d) \in \mathcal{D}$ and all $i \in N$,

$$\varphi_{R[i]}(N, d) \leq \frac{d_i + d_{i+1} + \cdots + d_n}{i}.$$

Our final property for polluted river problems is as follows: Imagine that state n pays the cost $\varphi_{R[n]}(N, d)$ and leaves. Furthermore, imagine that the remaining pollutant $d_n - \varphi_{R[n]}(N, d)$ is added to the pollutant caused by state $n-1$. As a result, the pollutant caused by state $n-1$ is adjusted to $d_{n-1} + d_n - \varphi_{R[n]}(N, d)$. The following property requires that the outcome the rule chooses for each problem should be invariant under the departure of the last-watercourse state. It is *self-dual.*

Last-watercourse state consistency: For all $(N, d) \in \mathcal{D}$ with $n \geq 2$, and all $i \in N'$,

$$\varphi_{R[i]}(N, d) = \varphi_{R[i]}(N', d'),$$

[14] Theorem 2 is not *dual* of Theorem B, since it does not deal with an axiomatization of the Shapley rule for bidding ring problems. However, this theorem is derived from the *duality* between the liability game and the river game, and from Remark 2-(ii). In this sense, it is *related dual* of Theorem B.

where $N' = N\backslash\{n\}$ and $d' = (d_1, d_2, \cdots, d_{n-2}, d_{n-1} + d_n - \varphi_{R[n]}(N, d)) \in \mathcal{D}$.

Theorem 3 *(Self-dual of Theorem 2) For polluted river problems, the Shapley rule is the only rule satisfying the equal share lower bound, the equal share upper bound, and last-watercourse state consistency.*

4.5 Anti-duality approach to claims problems and public good problems

As an application of the *anti-duality* approach to an analysis of rules for allocation problems, we consider claims problems and public good problems. We uncover the hidden relationship between the nucleolus rules for claims problems and for public good problems.

Imagine the situation where the liquidation value of a bankrupt firm has to be allocated between its creditors, but this resource cannot be jointly honored. The problem is to determine how to share the liquidation value (or, the endowment) between the creditors (O'Neill 1982).

A **claims problem** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$. The number c_i is the **claim** of creditor i on the **endowment** $\boldsymbol{E}$. An allocation, referred to as an **awards vector**, for (c, E) is a vector $x \in \mathbb{R}^N_+$ such that $\sum_N x_i = E$. Let $\boldsymbol{C^N}$ be the class of all claim problems for N.

Given $(c, E) \in C^N$, the **claims game** is defined by setting, for all $S \subseteq N$,

$$v_{CL}(c, E)(S) = \max\left\{0,\ E - \sum_{i \in N\backslash S} c_i\right\}.$$

For all $S \in 2^N$, the number $v_{CL}(c, E)(S)$ represents the difference between the endowment and the sum of claims of the creditors who form the complementary coalition $N\backslash S$ (or 0, if this difference is negative). This amount is conceded by the complementary coalition $N\backslash S$.

Let $\mathcal{V}_{CL}$ be the class of all claims games. Given $(c, E) \in C^N$, let $X(c, E)$ be the set of awards vectors for (c, E). A **solution for claims games** is a mapping $\phi_{CL} : \mathcal{V}_{CL} \to \mathbb{R}^N$ that associates with each claims game $v_{CL}(c, E)$ in the domain an awards vector in $X(c, E)$. We refer to the composite mapping $\varphi_{CL} \equiv \phi_{CL} \circ v_{CL}$ as a **rule for claims problems**. The **nucleolus rule for claims problems** is defined by $\varphi^{Nu}_{CL} \equiv Nu \circ v_{CL}$.

Concede-and-divide, denoted $\boldsymbol{CD(c, E)}$, is a rule for the two-claimant case. First, it assigns to each creditor $i \in N$ with $|N| = 2$ the difference $\max\{E - c_j, 0\}$ between the endowment and the other creditor's claim (or 0, if this difference is negative). This amount is conceded by the other creditor j. Next, this rule divides the remainder equally between them. As a result, for all $i \in N$,

$$CD_i(c,E) = \max\{E - c_j, 0\} + \frac{E - \sum_N \max\{E - c_k, 0\}}{2}.$$

A rule is **concede-and-divide consistent** if for all $N \in \mathcal{N}$, all $(c,E) \in C^N$, all $i,j \in N$ with $i \neq j$, if $x = \varphi(c,E)$, then $(x_i, x_j) = \varphi((c_i, c_j), x_i + x_j)$.

Theorem C *(Aumann and Maschler 1985) For claims problems, the nucleolus rule is the only rule that satisfies concede-and-divide consistency.*

Next, we consider the following "public good problems": Imagine the situation where a fixed size of a public good can be provided at a cost, and each agent consumes the public good. The social benefit is the difference between the sum of benefits of all agents and the cost of the public good. The problem is to determine how to share the social benefit between the agents.

A **public good problem** is a pair $(c,E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ such that $\sum_N c_i \geq E$. The number c_i is the **benefit** of agent i who consumes the public good. The public good can be provided at a **cost** $\boldsymbol{E}$. The social benefit is given by $\sum_N c_i - E$. An allocation for (c,E) is a vector $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = \sum_N c_i - E$. Let $\boldsymbol{C^N}$ be the class of all public good problems for N.

Given $(c,E) \in C^N$, the **public good game** is defined by setting, for all $S \subseteq N$,

$$v_{PG}(c,E)(S) = \max\left\{0,\ \sum_{i \in S} c_i - E\right\}.$$

For all $S \in 2^N$, the number $v_{PG}(c,E)(S)$ represents the difference between the sum of benefits of the agents who form the coalition S and the cost E (or 0, if this difference is negative).[15]

Let $\mathcal{V}_{PG}$ be the class of all public good games. Given $(c,E) \in C^N$, let $X(c,E)$ be the set of allocations for (c,E). A **solution for public good games** is the mapping $\phi_{PG} : \mathcal{V}_{PG} \to \mathbb{R}^N$ that associates with each public good game $v_{PG}(c,E)$ in the domain an allocation in $X(c,E)$. We refer to the composite mapping $\varphi_{PG} \equiv \phi_{PG} \circ v_{PG}$ as a **rule for public good problems**. The **nucleolus rule for public good problems** is defined by $\varphi_{PG}^{Nu} \equiv Nu \circ v_{PG}$.

Proposition B *(Oishi and Nakayama 2009) The following assertions hold:*

(i) Let w be the additive game given by setting, for all $S \subseteq N$, $w(S) \equiv \sum_S c_i$. Then,

$$v_{CL}^{ad} = v_{PG} - w, \text{ and } v_{PG}^{ad} = v_{CL} - w.$$

[15] The game v_{PG} can be reinterpreted as a "production game" (see Dutta and Ray 1989, pp.629-630).

(ii) $Nu(v_{CL}) = -Nu(v_{CL}^{ad})$*, and* $Nu(v_{PG}) = -Nu(v_{PG}^{ad})$*.*

As Oishi and Nakayama (2009, see pp.564) pointed out, Proposition B implies that for all $(c, E) \in C^N$, and all $i \in N$,

$$\begin{aligned}\varphi_{PG[i]}^{Nu}(c, E) &= Nu_{[i]}(v_{PG} - w + w) = Nu_{[i]}(v_{PG} - w) + c_i \\ &= -Nu_{[i]}(v_{CL}) + c_i = c_i - \varphi_{CL[i]}^{Nu}(c, E).\end{aligned}$$

Thus, we obtain the following result:

Proposition 4 *For all* $(c, E) \in C^N$*,* $\varphi_{PG}^{Nu}(c, E) = c - \varphi_{CL}^{Nu}(c, E)$*.*

The scenario underlying this proposition is as follows: In each public good problem, let us consider the situation where the agents have to share the cost E between themselves. This cost-sharing problem is described by a *claims problem* (c, E). Each agent i shares the cost $\varphi_{CL[i]}^{Nu}(c, E)$, and thus his net-benefit is $c_i - \varphi_{CL[i]}^{Nu}(c, E)$. Proposition 4 says that the each agent's outcome the nucleolus rule chooses for each public good problem is equal to the difference between his benefit and his cost the nucleolus rule chooses for the corresponding cost-sharing problem (i.e. the claims problem).

Remark 3 *The public good game* v_{PG} *is a claims game with the endowment* E *being replaced by* $\sum_N c_i - E$*. Given* $(c, E) \in C^N$*, for all* $S \subseteq N$*,* $v_{PG}(c, E)(S) = v_{CL}(c, \sum_N c_i - E)(S)$*. By this observation together with Proposition 4, we have that for all* $(c, E) \in C^N$*, and all* $i \in N$*,* $\varphi_{CL}^{Nu}(c, E) = c - \varphi_{CL}^{Nu}(c, \sum_N c_i - E)$*. This property says that the* ***Talmud rule*** *(Aumann and Maschler 1985) is self-dual in claims problems (see Thomson and Yeh 2008, pp.180).*[16]

Given $(c, E) \in C^N$ and a rule φ_{PG} for public good problems, we refer to $c - \varphi_{PG}(c, E)$ as the **cost allocation derived from a rule for public good problems**. By Theorem C and Proposition 4, we obtain the following result:

Theorem 4 *The cost allocation derived from the nucleolus rule for public good problems is the outcome the only concede-and-divide consistent rule chooses for the corresponding claims problems.*

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[16] The Talmud rule coincides with the nucleolus rule for claims problems.

Appendix

Proof of Proposition 2. Let φ be a *single-valued* solution on $\mathcal{V}_{vex}^{N}$ that satisfies *bilateral self-consistency*. Let $N \in \mathcal{N}$, v be a *convex* game for N, and $x \equiv \varphi^{ad}(v)$. By the definition of φ^{ad}, $x = -\varphi(-v^d)$. Let $N' \subset N$ with $|N'| = 2$, and $w \in \mathbb{R}^{2^{N'}}$ be such that for all $S \subseteq N'$,

$$w(S) = \begin{cases} -v^d(N) + \sum_{i \in N \setminus N'} x_i & \text{if } S = N', \\ -v^d\left(S \cup (N \setminus N')\right) - \sum_{i \in N \setminus N'} \varphi_i\left(-v^d\left|_{S \cup (N \setminus N')}\right.\right) & \text{if } S \neq N', \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

Since φ satisfies *bilateral self-consistency*, $w \in \mathcal{V}_{vex}^{N'}$ and $x_{N'} = -\varphi(w)$. Again by the definition of φ^{ad}, $x_{N'} = \varphi^{ad}(-w^d)$.

First, we have

$$-w^d(N') = v^d(N) - \sum_{i \in N \setminus N'} x_i = v(N) - \sum_{i \in N \setminus N'} \varphi_i^{ad}(v).$$

Next, for all $S \subset N'$ with $S \neq \emptyset$,

$$\begin{aligned} w(S) &= -v^d\left(S \cup (N \setminus N')\right) - \sum_{i \in N \setminus N'} \varphi_i\left(-v^d\left|_{S \cup (N \setminus N')}\right.\right) \\ &= -v(N) + v\left(N' \setminus S\right) - \sum_{i \in N \setminus N'} \varphi_i\left(-v^d\left|_{S \cup (N \setminus N')}\right.\right). \end{aligned}$$

Thus, for all $S \subset N'$ with $S \neq \emptyset$, we have

$$\begin{aligned} -w^d(S) &= -w(N') + w(N' \setminus S) \\ &= v(N) - \sum_{i \in N \setminus N'} x_i - v(N) + v\left(N' \setminus (N' \setminus S)\right) - \sum_{i \in N \setminus N'} \varphi_i\left(-v^d\left|_{(N' \setminus S) \cup (N \setminus N')}\right.\right) \\ &= -\sum_{i \in N \setminus N'} x_i + v(S) - \sum_{i \in N \setminus N'} \varphi_i\left(-v^d\left|_{N \setminus S}\right.\right), \end{aligned}$$

so that

$$-w^d(S) = v(S) + \sum_{i \in N \setminus N'} \varphi_i^{ad}\left(-\left(-v^d\left|_{N \setminus S}\right.\right)^d\right) - \sum_{i \in N \setminus N'} \varphi_i^{ad}(v).$$

Note that for all $T \subseteq N \setminus S$,

$$-v^d\big|_{N \setminus S}(T) = -v^d(T) = -v(N) + v(N \setminus T).$$

Thus, for all $T \subseteq N \backslash S$,

$$\begin{aligned}
-\left(-v^d\big|_{N\backslash S}\right)^d(T) &= -(-v^d\big|_{N\backslash S})(N\backslash S) + (-v^d\big|_{N\backslash S})\left((N\backslash S)\backslash T\right) \\
&= v(N) - v(N\backslash(N\backslash S)) - v(N) + v\left(N\backslash((N\backslash S)\backslash T)\right) \\
&= v(S \cup T) - v(S) \\
&= v^{N\backslash S}(T),
\end{aligned}$$

the desired conclusion. ■

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