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A Core Equivalence Theorem of an Assignment Market with Middlemen

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This study presents a new model of an assignment market with two types of middlemen: a classical type and a modern type. Classical middlemen such as Mediterranean traders in the Middle Ages open marketplaces between sellers and buyers, who incur transaction costs respectively. Modern middlemen such as real estate brokers in housing markets can eliminate transaction costs of sellers and buyers by matching them. The modern middlemen incur matching cost, whereas the classical middlemen do not. I show that under certain conditions of cost structure and of the number of the agents the set of all competitive equilibrium allocations of the assignment market with middlemen coincides with the core of the corresponding market game. This core equivalence theorem implies that there does not necessarily exist a competitive equilibrium with transaction via modern (resp. classical) middlemen even if the transaction costs of each seller and each buyer are sufficiently higher (resp. lower) than the matching cost of each modern middleman.

1 Introduction

Shapley and Shubik (1972) analyzed a competitive market for indivisible goods, namely an assignment market, from the viewpoint of game theory. In the assignment market, each seller owns one unit of indivisible goods initially and she wants to sell it; and each buyer wants to purchase at most one unit of the indivisible goods. Shapley and Shubik (1972) introduced a market game associated with the assignment market, and they showed that the core of this assignment game is never empty. Furthermore, they showed that the core coincides with the set of all competitive equilibrium allocations of the assignment market. These results imply that there always exists a competitive equilibrium in the assignment market; and the law of one price in the market holds without considering large coalitions in the economy. Many economists have applied the assignment game to analyses of markets with indivisible goods such as housing markets and labor markets, e.g., Shapley and Shubik (1972), Kaneko (1976, 1982), Crawford and Knoer (1981), and Kelso and Crawford (1982).

In the real world, there are also various assignment markets with middlemen, e.g., housing markets with real estate brokers and intermediate labor markets. Introducing models different from the assignment market model, in the existing literature, several researchers investigate markets for indivisible goods with middlemen, e.g., Rubinstein and Wolinsky (1987), Johri and Leach (2002), Yano (2008) and Blume et al. (2009). Rubinstein and Wolinsky (1987) and Johri and Leach

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(2002) investigated the activity of middlemen who play a role of matchmakers from the viewpoint of search theory. Blume et al. (2009) presented a trading network model in which each middleman is a market maker. Yano (2008) incorporated outside competitive forces of middlemen in his market bargaining model in order to deal with a certain fairness in an M&A market.

Although the existing literature gives some insight into the study of markets with middlemen, the following important questions may be still left for us: (i): Do middlemen yield Pareto-efficient outcomes in competitive equilibrium? What kind of roles of middlemen, if any, is related to the efficiency in the markets?; (ii): Does the law of one price hold in assignment markets with several types of middlemen? In order to deal with these questions, this paper aims to develop the Shapley and Shubik's assignment game framework. Using this new framework, I investigate the relationship between the core and the set of competitive equilibrium allocations associated with roles of middlemen. A research along this line would make us better understand the distributive function of the real markets with middlemen. In housing markets, for example, real estate brokers, who can eliminate transaction costs of house sellers and house buyers by matching them, make smooth transaction between the house sellers and the house buyers; so the role of real estate brokers is important in both efficiency of house allocations and price formation. Thus, the price mechanism associated with roles of middlemen is a central part serving the distribution of profits in markets with middlemen.

Toward the purpose of the present study, I introduce two types of middlemen and cost structure into the Shapley and Shubik's market model. The cost structure consists of transaction costs of both sellers and buyers, and matching costs of middlemen. Transaction costs[1)] are regarded as opportunity costs for transaction including (a): transportation costs, and (b): costs for measuring quality of goods. Transaction costs for measuring quality of goods may be justified by considering the situation in which each of market participants knows quality of goods exactly with his opportunity costs for measuring it. On the other hand, matching costs are regarded as opportunity costs for matching agents who search trading partners, respectively. The present study deals with a market in which all agents incur opportunity costs to perform transaction without time-delay.[2)]

In the present model, the following two types of middlemen are incorporated, namely, a classical type and a modern type. By the classical middlemen, I mean middlemen who open marketplaces between sellers and buyers. The underlying situation of the activity of the classical middlemen is the situation where there is no marketplace between sellers and buyers since they cannot trade freely because of difficulties of transport, politics or religion and so on. For example, classical middlemen may be regarded as the Maghribi traders in the eleventh-century (Grief, 1989), and as stallholders in the Makola marketplace in Ghana in the twentieth century (McMillan, 2002). A profit of a classical middleman is interpreted as a charge for the use of his marketplace by sellers and buyers. Unlike classical middlemen, the modern middlemen play a role of matchmakers between sellers and buyers. The underlying situation of the activity of the modern middlemen is the situation where the modern middlemen can eliminate transaction costs of sellers and buyers by matching them with his matching cost. A profit of a modern middleman is interpreted as a brokerage fee. The role of the modern middlemen can be found in real estate brokers and employment agencies, e.g., Rubinstein and Wolinsky (1987), Gehring (1993), and Yavaş (1992, 1994). Under the assumption of these simplified types of middlemen, it would be easy for us to understand how different roles of middlemen are related to the existence of a competitive equilibrium explicitly.

The main results may be summarized as follows. Consider the case where (i): the sum of the transaction costs of each seller and each buyer is relatively higher (resp. lower) than the matching cost of each modern middleman, and (ii): the number of modern (resp. classical) middlemen is not less than the number of potential assignments of goods. The number of potential assignments is given by the minimum of the number of sellers (goods) and the number of buyers. Then, the core of the corresponding market game (i.e. the three-sided assignment game) coincides with the set of all competitive equilibrium allocations of the market with transaction via modern (resp. classical) middlemen. In this case, a necessary and sufficient condition for the existence of a competitive equilibrium is that matching for transaction in the corresponding market is socially optimal. By the socially optimal matching for transaction, I mean that the matching for transaction yields the maximum of the social surplus in the market. The existence of the socially optimal matching for transaction is characterized by the existence of an integral solution for a certain linear program introduced by Quint (1991b). Unfortunately, it is known that there does not necessarily exist an integral solution for the Quint's linear program. These results have the following economic implications: First, in assignment markets with middlemen, the law of one price holds. Next, there does not necessarily exist a competitive equilibrium with transaction via modern (resp. classical) middlemen even if the transaction costs of each seller and each buyer are sufficiently higher (resp. lower) than the matching cost of each modern middleman. These implications are not derived from the results in Shapley and Shubik (1972).

The rest of this paper is organized as follows. I will introduce the assignment market model with the two types of middlemen and with cost structure in Section 2. Section 3 reports the main results of the core and competitive equilibria by using the three-sided assignment game. I will close this paper with concluding remarks in Section 4.

2 The model of a market with middlemen

First, I will introduce a model of a *market with middlemen.* This model is a generalization of the model proposed by Oishi and Sakaue (2009).[3]

Let $N_1=\{i_1,i_2,...,i_{n_1}\}$ and $N_3=\{k_1,k_2,...,k_{n_3}\}$ be the set of sellers and the set of buyers, respectively. I define the set of modern middlemen as $J^m=\{j_1^m, j_2^m,..., j_{n_m}^m\}$, and the set of classical middlemen as $J^c=\{j_1^c, j_2^c,..., j_{n_c}^c\}$. Let N_2 be the set of all middlemen, namely $N_2 = J^m \cup J^c$ and $|N_2| = n_2 = n_m + n_c$. Note that n_1, n_m, n_c, $n_3 \in \mathbb{N}$. The set of all agents is defined as $N = N_1 \cup N_2 \cup N_3$.

There are n_1 kinds of indivisible goods, and they are exchanged for money. Each seller $i \in N_1$ owns only one unit of indivisible goods initially, namely $\omega_i = 1$. Each middleman and each buyer own no unit of goods initially. Let us denote the demand and the supply of this market as follows.

Sellers' side: Each seller i takes one of the following three actions: (i): she sells one unit of her goods to a middleman in J^m; (ii): she sells it to a buyer via a middleman in J^c, with her transaction cost[4)] $c_i \geq 0$; (iii): she consumes her goods by herself.

Let x_i be the consumption of seller i, namely $x_i \in \{0, 1\}$.

Middlemen's side: Each middleman j in N_2 wants to sell at most one unit of goods to only a buyer. For this purpose, each middleman purchases at most one unit of goods from only a seller. Let the matching cost[5)] of each middleman $j \in N_2$ be given by $c_{j^m} \geq 0$ for each $j^m \in J^m$ and $c_{j^c} = 0$ for each $j^c \in J^c$. Note that each classical middleman does not incur matching cost since he does not match a seller and a buyer. Let $\tilde{x}_{ij}$ be the supply of seller i to middleman $j \in N_2$, namely $\tilde{x}_{ij} \in \{0, 1\}$. I assume that each middleman $j \in N_2$ consumes no unit of goods which he purchases from the seller.

Buyers' side: Each buyer k takes one of the following two actions in order to consume at most one unit of goods: (i): she purchases at most one unit of goods from only a middleman in J^m; (ii): she purchases at most one unit of goods from a seller via a middleman in J^c with her transaction cost[6)] $c_k \geq 0$. Let x_{ijk} be the consumption of buyer k, namely $x_{ijk} \in \{0, 1\}$. In the case of $x_{ijk} = 1$, buyer k demands one unit of goods which middleman $j \in N_2$ purchases from seller i. Moreover, let $\tilde{x}_{ijk}$ be the supply of middleman $j \in N_2$, namely $\tilde{x}_{ijk} \in \{0, 1\}$. In the case of $\tilde{x}_{ijk} = 1$, middleman $j \in N_2$ supplies to buyer k one unit of goods which the middleman purchases from seller i.

A1, **A2**, and **A3** give the set of feasible allocations of sellers, middlemen and buyers, respectively.

A1 : For all $i \in N_1$, $X_i \equiv \{(x_i, (\tilde{x}_{ij})_{j \in N_2}) \in \mathbb{Z}_+^{1+n_2} : x_i + \sum_{j \in N_2} \tilde{x}_{ij} = \omega_i = 1\}$.

A2 : For all $j \in N_2$, $X_j \equiv \{(\tilde{x}_{ijk})_{i \in N_1, k \in N_3} \in \mathbb{Z}_+^{n_1 n_3} : \sum_{i \in N_1} \sum_{k \in N_3} \tilde{x}_{ijk} \leq 1\}$.

A3 : For all $k \in N_3$, $X_k \equiv \{(x_{ijk})_{i \in N_1, j \in N_2} \in \mathbb{Z}_+^{n_1 n_2} : \sum_{i \in N_1} \sum_{j \in N_2} x_{ijk} \leq 1\}$.

Each seller and each buyer have utility functions on consumption. Any utility is measured in terms of money. These functions of each seller and each buyer are given by $U_i : \mathbb{Z}_+ \to \mathbb{R}$ for all $i \in N_1$ and $U_k : \mathbb{Z}_+^{n_1 n_2} \to \mathbb{R}$ for all $k \in N_3$, respectively. The utility functions $U_i(\cdot)$ and $U_k(\cdot)$ are non-decreasing. Assume $U_i(0) = 0$ and $U_k(\mathbf{0}) = 0$, where $\mathbf{0} \in \mathbb{Z}_+^{n_1 n_2}$.

Each middleman j in N_2 purchases at most one unit of goods at a price $p_{ij} \in \mathbb{R}_+$ from seller i. Also, middleman j in N_2 sells at most one unit of i's initial goods at a price $q_{ijk} \in \mathbb{R}_+$ to buyer k. Let two distinct price lists be given by $p = (p_{ij})_{i \in N_1, j \in N_2} \in \mathbb{R}_+^{n_1 n_2}$ and $q = (q_{ijk})_{i \in N_1, j \in N_2, k \in N_3} \in \mathbb{R}_+^{n_1 n_2 n_3}$ respectively.

Let the mapping $\mathbf{1}_m : N_2 \to \{0, 1\}$ be given by (1): $j \in J^m \mapsto 1$, and (2) : $j \in J^c \mapsto 0$. Similarly, let the mapping $\mathbf{1}_c : N_2 \to \{0, 1\}$ be given by (1) : $j \in J^c \mapsto 1$, and (2): $j \in J^m \mapsto 0$.

Then, utility outcomes of all agents are given by the followings.

- For all $i \in N_1$, $U_i(x_i) + \sum_{j \in N_2} p_{ij} \tilde{x}_{ij} - c_i(\sum_{j \in N_2} \tilde{x}_{ij} \mathbf{1}_c(j))$.
- For all $j \in N_2$, $-\sum_{i \in N_1} p_{ij}(\sum_{k \in N_3} \tilde{x}_{ijk}) + \sum_{i \in N_1} \sum_{k \in N_3} q_{ijk} \tilde{x}_{ijk} - \sum_{i \in N_1} c_j (\sum_{k \in N_3} \tilde{x}_{ijk} \mathbf{1}_m(j))$.
- For all $k \in N_3$, $U_k((x_{ijk})_{i \in N_1, j \in N_2}) - \sum_{i \in N_1} \sum_{j \in N_2} q_{ijk} x_{ijk} - \sum_{i \in N_1} \sum_{j \in N_2} c_k x_{ijk} \mathbf{1}_c(j)$.

It would not be necessary to explain the meaning of utility outcomes of each seller and each buyer, respectively. Here, the meaning of the utility outcome of each middleman will be given as follows: This utility outcome consists of three components. The first component is the total cost in his purchasing goods from a seller. The second component is the revenue in his selling goods to a buyer. The third component is matching costs, if he is a modern middleman. If he is a classical middleman, the third component is nothing.

Next, **A4** and **A5** give the market-clearing conditions as follows.

A4: For all $(i, j) \in N_1 \times N_2, \Sigma_{k \in N_3} \tilde{x}_{ijk} = \tilde{x}_{ij}$.

A5: For all $(i, j, k) \in N_1 \times N_2 \times N_3, x_{ijk} = \tilde{x}_{ijk}$.

A tuple $(\hat{p}, \hat{q}, \hat{x}) = ((\hat{p}_{ij})_{i \in N_1, j \in N_2}, (\hat{q}_{ijk})_{i \in N_1, j \in N_2, k \in N_3}, ((\hat{x}_i)_{i \in N_1}, (\hat{x}_{ijk})_{i \in N_1, j \in N_2, k \in N_3})) \in \mathbb{R}_+^{n_1 n_2} \times \mathbb{R}_+^{n_1 n_2 n_3} \times \mathbb{Z}_+^{n_1 + n_1 n_2 n_3}$ is called a *competitive equilibrium* if $(\hat{p}, \hat{q}, \hat{x})$ satisfies

(**I**): for all $i \in N_1, U_i(\hat{x}_i) + \max_{j \in N_2} (\hat{p}_{ij} - c_i \mathbf{1}_c(j))(\omega_i - \hat{x}_i) = \max_{(x_i, (\tilde{x}_{ij})_{j \in N_2}) \in X_i} [U_i(x_i) + \Sigma_{j \in N_2} (\hat{p}_{ij} - c_i \mathbf{1}_c(j)) \tilde{x}_{ij}]$,

(**II**): for all $j \in N_2, \Sigma_{i \in N_1} \{ \Sigma_{k \in N_3} \hat{q}_{ijk} \hat{x}_{ijk} - (\hat{p}_{ij} + c_j \mathbf{1}_m(j))(\Sigma_{k \in N_3} \hat{x}_{ijk}) \} = \max_{(\tilde{x}_{ijk})_{i \in N_1, k \in N_3} \in X_j} [\Sigma_{i \in N_1} \{ \Sigma_{k \in N_3} \hat{q}_{ijk} \tilde{x}_{ijk} - (\hat{p}_{ij} + c_j \mathbf{1}_m(j))(\Sigma_{k \in N_3} \tilde{x}_{ijk}) \}]$,

(**III**): for all $k \in N_3, U_k((\hat{x}_{ijk})_{i \in N_1, j \in N_2}) - \Sigma_{i \in N_1} \Sigma_{j \in N_2} (c_k \mathbf{1}_c(j) + \hat{q}_{ijk}) \hat{x}_{ijk} = \max_{(x_{ijk})_{i \in N_1, j \in N_2} \in X_k} [U_k((x_{ijk})_{i \in N_1, j \in N_2}) - \Sigma_{i \in N_1} \Sigma_{j \in N_2} (c_k \mathbf{1}_c(j) + \hat{q}_{ijk}) x_{ijk}]$, and

(**IV**): for all $i \in N_1, \hat{x}_i + \Sigma_{j \in N_2} \Sigma_{k \in N_3} \hat{x}_{ijk} = \omega_i (=1)$.

Note that (**IV**) is equivalent to **A1** through **A5**. This means the equilibrium conditions for all goods.

A *competitive equilibrium price* is a pair $(\hat{p}, \hat{q})$ if there exists a competitive equilibrium $(\hat{p}, \hat{q}, \hat{x})$. If there exists a competitive equilibrium $(\hat{p}, \hat{q}, \hat{x})$, then a *competitive outcome* $(\hat{u}, \hat{v}, \hat{w}) \in \mathbb{R}^{n_1 + n_2 + n_3}$ is given by the followings.

- For all $i \in N_1, \hat{u}_i = U_i(\hat{x}_i) + \max_{j \in N_2} (\hat{p}_{ij} - c_i \mathbf{1}_c(j))(\omega_i - \hat{x}_i)$.
- For all $j \in N_2, \hat{v}_j = \Sigma_{i \in N_1} \{ \Sigma_{k \in N_3} \hat{q}_{ijk} \hat{x}_{ijk} - (\hat{p}_{ij} + c_j \mathbf{1}_m(j))(\Sigma_{k \in N_3} \hat{x}_{ijk}) \}$.
- For all $k \in N_3, \hat{w}_k = U_k((\hat{x}_{ijk})_{i \in N_1, j \in N_2}) - \Sigma_{i \in N_1} \Sigma_{j \in N_2} (c_k \mathbf{1}_c(j) + \hat{q}_{ijk}) \hat{x}_{ijk}$.

3 The core and competitive equilibria

Let $|N| = n$ and let π be the subset of 2^N given by $\pi \equiv \{\{i\} : i \in N\} \cup \{\{i, j, k\} : i \in N_1, j \in N_2, k \in N_3\}$. Note that π stands for the matching structure among the agents in the three-sided assignment market. Let $S \subseteq N$ be a coalition. A π-partition of S, ρs is defined as any partition of S into π. Let Ps be the class of all π-partitions of S.

Let $a = (a_T)_{T \in \pi} \in \mathbb{R}_+^{n + n_1 n_2 n_3}$ be the vector satisfying (1): $a_{\{i\}} = U_i(\omega_i)$ for all $i \in N_1$ and $a_{\{j\}} = a_{\{k\}} = 0$ for all $j \in N_2$ and all $k \in N_3$, and (2): $a_{\{i, j, k\}} = U_k(e^{ij}) - c_i \mathbf{1}_c(j) - c_j \mathbf{1}_m(j) - c_k \mathbf{1}_c(j)$ for all $(i, j, k) \in N_1 \times N_2 \times N_3$, where e^{ij} is the $n_1 n_2$-dimensional vector such that $e^{ij}_{ij} = 1$ and $e^{ij}_{i'j'} = 0$ for all $(i', j') \in N_1 \times N_2$ with $(i', j') \neq (i, j)$. Note that $U_k(e^{ij})$ stands for the utility outcome of buyer k if she consumes seller i's goods via middleman $j \in N_2$. The meaning of $a_{\{i, j, k\}}$ is the social surplus yielded by transaction in which buyer k consumes seller i's good via middleman $j \in N_2$. For the sake of simplicity, I assume that $a_{\{i, j, k\}}$ is non-negative for all $(i, j, k) \in N_1 \times N_2 \times N_3$.

Following Kaneko and Wooders (1982), I can give the *three-sided assignment game* as (N, V), where

$$V(S) \equiv \max_{\rho s \in Ps} \sum_{T \in \rho s} a_T \text{ for nonempty } S \subseteq N \text{ with } V(\emptyset) = 0,$$

and the *core* of (N, V) as the set of utility vectors $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^n$ satisfying (1): $\Sigma_{i \in N_1} \bar{u}_i + \Sigma_{j \in N_2} \bar{v}_j + \Sigma_{k \in N_3} \bar{w}_k = V(N)$, and (2): for all $T \in \pi$, $\Sigma_{i \in N_1 \cap T} \bar{u}_i + \Sigma_{j \in N_2 \cap T} \bar{v}_j + \Sigma_{k \in N_3 \cap T} \bar{w}_k \geq V(T)$. $V(S)$ means the social surplus yielded by transaction among the members in S. In the definition of the core, condition (1) is the efficiency condition, and condition (2) is the stability conditions.

Lemma 1 *Each competitive outcome belongs to the core of* (N, V).

Proof. Let $(\hat{u}, \hat{v}, \hat{w})$ be a competitive outcome. Then, there exists a competitive equilibrium $(\hat{p}, \hat{q}, \hat{x})$ which attains $(\hat{u}, \hat{v}, \hat{w})$.

Let S be a subset in N. Let ρ_S^* be a π-partition which attains $V(S)$. Let $z = ((z_i)_{i\in N_1}, (z_{ijk})_{i\in N_1, j\in N_2, k\in N_3})$ be the vector in $\mathbb{Z}_+^{n_1+n_1n_2n_3}$ satisfying

(i): $z_i = 1$ if $\{i\} \in \rho_S^*$ such that $i \in N_1$,
(ii): $z_i = 0$ if $\{i\} \notin \rho_S^*$ such that $i \in N_1 \cap S$,
(iii): $z_i = 1$ if $i \in N_1 \setminus S$,
(iv): $z_{ijk} = 1$ if $\{i, j, k\} \in \rho_S^*$ such that $(i, j, k) \in N_1 \times N_2 \times N_3$; and
(v): $z_{ijk} = 0$ otherwise.

Step 1: Let $z_i = x_i$, $z_{ijk} = x_{ijk} = \tilde{x}_{ijk}$ and $\Sigma_{k\in N_3} z_{ijk} = \tilde{x}_{ij}$. Then I will prove that z satisfies **A1** through **A3**, namely the followings.

$$z_i + \sum_{j\in N_2}\sum_{k\in N_3} z_{ijk} = 1 \text{ for all } i \in N_1 \cap S. \quad (1)$$

$$\sum_{i\in N_1}\sum_{k\in N_3} z_{ijk} \leq 1 \text{ for all } j \in N_2 \cap S. \quad (2)$$

$$\sum_{i\in N_1}\sum_{j\in N_2} z_{ijk} \leq 1 \text{ for all } k \in N_3 \cap S. \quad (3)$$

First, formula (1) will be shown as follows:

Case 1: Suppose $\{\hat{\imath}\} \in \rho_S^*$ such that $\hat{\imath} \in N_1$. By (i) and (v), $z_{\hat{\imath}} = 1$ and $z_{\hat{\imath}jk} = 0$ for all $(j, k) \in N_2 \times N_3$. Then, $z_{\hat{\imath}} + \Sigma_{j\in N_2}\Sigma_{k\in N_3} z_{\hat{\imath}jk} = 1$.

Case 2: Suppose $\{\bar{\imath}\} \notin \rho_S^*$ such that $\bar{\imath} \in N_1 \cap S$. This implies that there exists a pair $(\bar{j}, \bar{k}) \in N_2 \times N_3$ such that $\{\bar{\imath}, \bar{j}, \bar{k}\} \in \rho_S^*$. By (iv) and (v), $z_{\bar{\imath}\bar{j}\bar{k}} = 1$ and $z_{\bar{\imath}jk} = 0$ for all $(j, k) \in N_2 \times N_3$ with $(j, k) \neq (\bar{j}, \bar{k})$. Also, by (ii), $z_{\bar{\imath}} = 0$. Then, $z_{\bar{\imath}} + \Sigma_{j\in N_2}\Sigma_{k\in N_3} z_{\bar{\imath}jk} = 1$. This completes the proof of formula (1).

Next, formula (2) will be shown as follows:

Case 1: Suppose $\{\hat{\jmath}\} \in \rho_S^*$ such that $\hat{\jmath} \in N_2$. By (v), $z_{i\hat{\jmath}k} = 0$ for all $(i, k) \in N_1 \times N_3$. Then, $\Sigma_{i\in N_1}\Sigma_{k\in N_3} z_{i\hat{\jmath}k} = 0$.

Case 2: Suppose $\{i', j', k'\} \in \rho_S^*$ such that $(i', j', k') \in N_1 \times N_2 \times N_3$. By (iv) and (v), $z_{i'j'k'} = 1$ and $z_{ij'k} = 0$ for all $(i, k) \in N_1 \times N_3$ with $(i, k) \neq (i', k')$. Then, $\Sigma_{i\in N_1}\Sigma_{k\in N_3} z_{ij'k} = 1$. This completes the proof of formula (2).

The proof of formula (3) is the same manner as the proof of formula (2).

Step 2: I will prove that

$$\sum_{i\in S\cap N_1} \hat{u}_i + \sum_{j\in S\cap N_2} \hat{v}_j + \sum_{k\in S\cap N_3} \hat{w}_k \geq V(S) \text{ for all } S \subseteq N.$$

By calculation,

$$\begin{aligned}
&\sum_{i\in S\cap N_1} \hat{u}_i + \sum_{j\in S\cap N_2} \hat{v}_j + \sum_{k\in S\cap N_3} \hat{w}_k \\
&\geq \sum_{i\in S\cap N_1} [U_i\ (z_i) + \sum_{j\in S\cap N_2}\sum_{k\in S\cap N_3} (\hat{p}_{ij} - c_i\mathbf{1}_c(j))z_{ijk}] \\
&\quad + \sum_{j\in S\cap N_2} [\sum_{i\in S\cap N_1} \{\sum_{k\in S\cap N_3} \hat{q}_{ijk} z_{ijk} - (\hat{p}_{ij} + c_j\mathbf{1}_m(j))(\sum_{k\in S\cap N_3} z_{ijk})\}] \\
&\quad + \sum_{k\in S\cap N_3} [U_k((z_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in S\cap N_1}\sum_{j\in S\cap N_2} (c_k\mathbf{1}_c(j) + \hat{q}_{ijk})z_{ijk}] \\
&= \sum_{i\in S\cap N_1} U_i(z_i) + \sum_{k\in S\cap N_3} [U_k(z_{ijk})_{i\in N_1, j\in N_2}) - \sum_{i\in S\cap N_1}\sum_{j\in S\cap N_2} c_i\mathbf{1}_c(j)z_{ijk} \\
&\quad - \sum_{i\in S\cap N_1}\sum_{j\in S\cap N_2} c_j\mathbf{1}_m(j)z_{ijk} - \sum_{i\in S\cap N_1}\sum_{j\in S\cap N_2} c_k\mathbf{1}_c(j)z_{ijk}] \\
&= \sum_{j\in\rho_S^*} a_{\{i\}} + \sum_{\{i,j,k\}\in\rho_S^*} a_{\{i,j,k\}} = V(S).
\end{aligned}$$

Note that the inequality holds because of Step 1 and the definition of the competitive equilibrium, and the second equality holds because of the definition of z.

Step 3: Since $(\hat{u}, \hat{v}, \hat{w})$ is the competitive outcome, $\Sigma_{i\in N_1}\hat{u}_i + \Sigma_{j\in N_2}\hat{v}_j + \Sigma_{k\in N_3}\hat{w}_k = V(N)$. This completes the proof of Lemma 1. ■

Next, consider a *partitioning linear program* proposed by Quint (1991b). Let $y = (y_T)_{T\in\pi} \in \mathbb{R}^{n+n_1n_2n_3}$ be a vector of control variables. Let $b = (b_i)_{i\in N} \in \mathbb{R}^n$ be a vector of control variables of a dual problem. The primal problem (P) for the partitioning linear program is defined as

$$(P)\ :\ \max_{(y_T)_{T\in\pi}} \sum_{T\in\pi} a_T y_T$$
$$s.t. \sum_{T\in\pi,\, T\ni i} y_T = 1 \text{ for all } i\in N$$
$$y_T \geq 0 \text{ for all } T\in\pi.$$

The dual problem of (P) is defined as

$$(D)\ :\ \min_{(b_i)_{i\in N}} \sum_{i\in N} b_i$$
$$s.t. \sum_{i\in T} b_i \geq a_T \text{ for all } T\in\pi.$$

The following theorem is proved by Quint (1991b).

Quint's Theorem (i): If (P) solves integrally (i.e. with all 0's and 1's), the core of the game (N, V) is nonempty and it coincides with the set of optimal solutions to (D); and (ii): if (P) does not solve integrally, the core of (N, V) is empty.

The potential market value is referred to as max $\Sigma_{T\in\pi} a_T y_T$ for $y_T \in \mathbb{Z}_+^{|\pi|}$ satisfying the constraints in (P). This terminology follows from Yang (2003). The potential market value is the maximum of the social surplus yielded by market transaction in N. When the partitioning linear program (P) has an integral solution with its value equal to the potential market value, I say that matching for transaction in the market is *socially optimal*. This is because matching for transaction expressed by the integral solution attains the potential market value. Quint's theorem states that socially optimal matching for transaction is a necessary and sufficient condition for non-emptiness of the core of the three-sided assignment game.

If there exists at least one integral solution y^* for (P), let $\hat{y}$ be the vector in $\mathbb{Z}_+^{n_1+n_1n_2n_3}$ such that $\hat{y}_i \equiv y^*_{\{i\}}$ for all $i \in N_1$ and $\hat{y}_{ijk} \equiv y^*_{\{i,j,k\}}$ for all $\{i, j, k\} \in \pi$. Let $Y \subseteq \mathbb{Z}_+^{n_1+n_1n_2n_3}$ be the set of all $\hat{y}$. Let D be the set of solutions of (D).

Let C^* be given by the set of utility vectors $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^n$ satisfying

(1): the π-partition efficiency conditions: for each $\hat{y}$ in Y,

$\bar{u}_i + \bar{v}_j + \bar{w}_k = a_{\{i,j,k\}}$ if $\hat{y}_{ijk} = 1$;

$\bar{u}_i = a_{\{i\}}$ if $\hat{y}_i = 1$;

(2): the stability conditions:

$\bar{u}_i + \bar{v}_j + \bar{w}_k \geq a_{\{i,j,k\}}$ if $\{i, j, k\} \in \pi$;

$\bar{u}_i \geq a_{\{i\}}$ if $i \in N_1$;

$\bar{v}_j \geq a_{\{j\}} = 0$ if $j \in N_2$;

$\bar{w}_k \geq a_{\{k\}} = 0$ if $k \in N_3$.

Next, I will state Lemmas 2 and 3. They are useful for the proof of Lemma 4. The proof of Lemmas 2 and 3 is the same manner as the proof of Lemmas 2 and 3 in Oishi and Sakaue (2009).

Lemma 2 *The core of (N, V) is a subset of C^*.*

Proof. Before beginning the proof, I shall give the complementary slackness condition (Dantzig, 1963, pp.135-136): Let y be a vector which satisfies the constraints of (P). Let b be a vector which satisfies the constraints of (D). Then y is a solution of (P) and b is a solution of (D) if and only if $\Sigma_{T\in\pi} y_T (\Sigma_{i\in T} b_i - a_T) = 0$. Using the complementary slackness condition, I will show this lemma as follows: Suppose that the core is nonempty. There exists a $\hat{y}$ in Y, which is derived from an integral solution y^* for (P). Moreover, D coincides with the core of the game (N, V). Choose any $(u', v', w') \in D$ and fix it. It is sufficient to prove that (u', v', w') satisfies (i): $u'_i+v'_j+w'_k = a_{\{i,j,k\}}$ if $\hat{y}_{ijk} = 1$ and (ii): $u'_i = a_{\{i\}}$ if $\hat{y}_i = 1$. By the complementary slackness condition, $u'_i+v'_j+w'_k = a_{\{i,j,k\}}$ if $\hat{y}_{ijk} = 1$ and $u'_i = a_{\{i\}}$ if $\hat{y}_i = 1$, which completes the proof. ■

Lemma 3 *Let $\hat{y} \in Y$. Fix any $j \in N_2$ and any $k \in N_3$.*

If $\hat{y}_{i'jk'} = 0$ *for all* $(i', k') \in N_1 \times N_3$,
then $\bar{v}_j = 0$;
if $\hat{y}_{i'j'k} = 0$ *for all* $(i', j') \in N_1 \times N_2$,
then $\bar{w}_k = 0$.

Proof. Let $\tilde{J} \equiv \{j \in N_2 : \hat{y}_{ijk} = 0$ for all $i \in N_1$ and all $k \in N_3\}$ and $\tilde{K} \equiv \{k \in N_3 : \hat{y}_{ijk} = 0$ for all $i \in N_1$ and all $j \in N_2\}$. Since $(\bar{u}, \bar{v}, \bar{w}) \in C^*$, $\bar{v}_j \geq 0$ for all $j \in \tilde{J}$ and $\bar{w}_k \geq 0$ for all $k \in \tilde{K}$. By Quint's theorem and Lemma 2 in the present study,

$$\sum_{j \in \tilde{J}} \bar{v}_j + \sum_{k \in \tilde{K}} \bar{w}_k = 0,$$

which implies $\bar{v}_j = 0$ for all $j \in \tilde{J}$ and $\bar{w}_k = 0$ for all $k \in \tilde{K}$. ■

Before Lemma 4, I will introduce the premium of each middleman. Let $\alpha_{ik}^{j^m j^c} \equiv U_k(e^{ij^m}) - U_k(e^{ij^c})$ for all $(i,k) \in N_1 \times N_3$ and all $(j^m, j^c) \in J^m \times J^c$. When $\alpha_{ik}^{j^m j^c} > 0$, $\alpha_{ik}^{j^m j^c}$ may be regarded as the premium of a modern middleman. When $\alpha_{ik}^{j^m j^c} < 0$, on the other hand, $-\alpha_{ik}^{j^m j^c}$ may be regarded as the premium of a classical middleman.

Lemma 4 (i): *Let* $\alpha_{ik}^{j^m j^c} + c_i + c_k \geq c_{j^m}$ *for all* $(i,k) \in N_1 \times N_3$ *and all* $(j^m, j^c) \in J^m \times J^c$, *and* $n_m \geq \min\{n_1, n_3\}$. *The core of* (N, V) *belongs to the set of competitive outcomes of the market with transaction via modern middlemen.* (ii): *Let* $\alpha_{ik}^{j^m j^c} + c_i + c_k \leq c_{j^m}$ *for all* $(i, k) \in N_1 \times N_3$ *and all* $(j^m, j^c) \in J^m \times J^c$, *and* $n_c \geq \min\{n_1, n_3\}$. *The core of* (N, V) *belongs to the set of competitive outcomes of the market with transaction via classical middlemen.*

Proof. I will prove (i) in this lemma. The proof of (ii) is the same manner as the proof of (i), hence it will be omitted. The statement in (i) is equivalent to the followings: Fix an arbitrary pair $(i, k) \in N_1 \times N_3$. Let $a_{\{i, j^m, k\}} \geq a_{\{i, j^c, k\}}$ for all $(j^m, j^c) \in J^m \times J^c$, and $n_m \geq \min\{n_1, n_3\}$. Then, the core of (N, V) belongs to the set of competitive outcomes of the market with transaction via modern middlemen.

Suppose that the core of (N, V) is nonempty. Then, Y is nonempty. Let $\hat{y} \in \{0, 1\}^{n_1 + n_1 n_2 n_3}$ be a vector in Y. Choose any $(\bar{u}, \bar{v}, \bar{w})$ in the core of (N, V).

Consider a price list $(\hat{p}, \hat{q})$ satisfying $\hat{p}_{ij^m} = \bar{u}_i$ for all $(i, j^m) \in N_1 \times J^m$, $\hat{p}_{ij^c} = \bar{u}_i + c_i$ for all $(i, j^c) \in N_1 \times J^c$, $\hat{q}_{ij^m k} = \bar{u}_i + \bar{v}_{j^m} + c_{j^m}$ for all $(i, j^m, k) \in N_1 \times J^m \times N_3$ and $\hat{q}_{ij^c k} = \bar{u}_i + \bar{v}_{j^c} + c_i$ for all $(i, j^c, k) \in N_1 \times J^c \times N_3$. I will prove that $(\hat{p}, \hat{q}, \hat{y})$ is a competitive equilibrium yielding $(\bar{u}, \bar{v}, \bar{w})$. Let $\hat{y}_i = x_i^*$, $\hat{y}_{ijk} = x_{ijk}^* = \tilde{x}_{ijk}^*$ and $\sum_{k \in N_3} \hat{y}_{ijk} = \tilde{x}_{ij}^*$. The last two formulas follow from **A4** and **A5**.

Step 1: $((x_i^*)_{i \in N_1}, (\tilde{x}_{ij}^*)_{i \in N_1, j \in N_2})$ satisfies **A1**, $(\tilde{x}_{ijk}^*)_{i \in N_1, j \in N_2, k \in N_3}$ satisfies **A2** and $(x_{ijk}^*)_{i \in N_1, j \in N_2, k \in N_3}$ satisfies **A3**, respectively. The proof will be omitted since it is a matter of calculation. Hence, condition (IV) is satisfied.

Step 2: I will show that for all $i \in N_1$ $(x_i^*, (\tilde{x}_{ij}^*)_{j \in N_2})$ is a maximal solution of (I), for all $j \in N_2$ $(\tilde{x}_{ijk}^*)_{i \in N_1, k \in N_3}$ is a maximal solution of (II), and for all $k \in N_3$ $(x_{ijk}^*)_{i \in N_1, j \in N_2}$ is a maximal solution of (III), respectively.

Substep 2.1: Fix an arbitrary seller $i \in N_1$. The seller i can obtain $a_{\{i\}}$ by consuming her own goods, or she can obtain $\hat{p}_{ij^m}$ or $\hat{p}_{ij^c} - c_i$ by selling her own goods to a middleman in N_2.

Case 1: Consider that $x_i^* = 1$ and $\tilde{x}_{ij}^* = 0$ for all $j \in N_2$. In this case, $a_{\{i\}} \geq \hat{p}_{ij^m}$ for all $j^m \in J^m$ and $a_{\{i\}} \geq \hat{p}_{ij^c} - c_i$ for all $j^c \in J^c$ must be satisfied. It can be checked since $a_{\{i\}} = \bar{u}_i = \hat{p}_{ij^m} = \hat{p}_{ij^c} - c_i$.

Case 2: Consider that $x_i^* = 0$ and there exists a $j^m \in J^m$ such that $\tilde{x}_{ij^m}^* = 1$ and $\tilde{x}_{ij'}^* = 0$ for all $j' \in N_2 \setminus \{j^m\}$. In this case, it must be satisfied that $\hat{p}_{ij^m} \geq a_{\{i\}}$, $\hat{p}_{ij^m} \geq \hat{p}_{ij''}$ for all $j'' \in J^m \setminus \{j^m\}$ and $\hat{p}_{ij^m} \geq \hat{p}_{ij^c} - c_i$ for all $j^c \in J^c$. By Lemma 2, it can be checked since $\hat{p}_{ij^m} = \bar{u}_i$

$\geq a_{\{i\}}$, $\hat{p}_{ij^m} = \bar{u}_i = \hat{p}_{ij''}$ for all $j'' \in J^m \setminus \{j^m\}$ and $\hat{p}_{ij^m} = \bar{u}_i = \hat{p}_{ij^c} - c_i$ for all $j^c \in J^c$.

Substep 2.2: Fix an arbitrary middleman $j^m \in J^m$. The middleman j^m can obtain $\hat{q}_{ij^mk} - \hat{p}_{ij^m} - c_{j^m}$ by buying one unit of goods from a seller in N_1 and selling it to a buyer in N_3, or he obtains 0 by doing nothing.

Case 1: Consider that $\tilde{x}^*_{ijk} = 0$ for all $i \in N_1$ and all $k \in N_3$. In this case, $0 \geq \hat{q}_{ij^mk} - \hat{p}_{ij^m} - c_{j^m}$ must be satisfied. It can be checked since $0 = \bar{v}_{j^m} = (\bar{u}_i + \bar{v}_{j^m}) - \bar{u}_i = \hat{q}_{ij^mk} - \hat{p}_{ij^m} - c_{j^m}$. Note that $\bar{v}_{j^m} = 0$ because of Lemma 3.

Case 2: Consider that there exists a pair $(i, k) \in N_1 \times N_3$ such that $\tilde{x}^*_{ij^mk} = 1$ and $\tilde{x}^*_{i'j^mk'} = 0$ for all $(i', k') \in N_1 \times N_3$ with $(i', k') \neq (i, k)$. In this case, it must be satisfied that $\hat{q}_{ij^mk} - \hat{p}_{ij^m} - c_{j^m} \geq \hat{q}_{i'j^mk'} - \hat{p}_{i'j^m} - c_{j^m}$ for all $i' \in N_1 \setminus \{i\}$ and all $k' \in N_3 \setminus \{k\}$, and $\hat{q}_{ij^mk} - \hat{p}_{ij^m} - c_{j^m} \geq 0$. It can be checked since $\hat{q}_{ij^mk} - \hat{p}_{ij^m} - c_{j^m} = (\bar{u}_i + \bar{v}_{j^m}) - \bar{u}_i = \bar{v}_{j^m} = (\bar{u}_{i'} + \bar{v}_{j^m}) - \bar{u}_{i'} = \hat{q}_{i'j^mk'} - \hat{p}_{i'j^m} - c_{j^m}$. Also, by Lemma 2, $\bar{v}_{j^m} \geq 0$.

Substep 2.3: Fix an arbitrary middleman $j^c \in J^c$. The middleman j^c always obtains 0. By Lemma 3 and the assumption of (*i*) in Lemma 4, $\bar{v}_{j^c} = 0$ for all $j^c \in J^c$. Therefore, $\hat{q}_{ij^ck} - \hat{p}_{ij^c} = \bar{v}_{j^c} = 0$ for all $(i, k) \in N_1 \times N_3$. This means that the middleman j^c always obtains 0.

Substep 2.4: Fix an arbitrary buyer $k \in N_3$. The buyer k can obtain $U_k(e^{ij^m}) - \hat{q}_{ij^mk}$ or $U_k(e^{ij^c}) - \hat{q}_{ij^ck} - c_k$ by buying one unit of goods of a seller in N_1 from a middleman in N_2, or she obtains 0 by consuming nothing. Since $x^*_{ij^ck} = \hat{y}_{ij^ck} = 0$ for all $(i, j^c, k) \in N_1 \times J^c \times N_3$, I will consider the following two cases.

Case 1: Consider that $x^*_{ijk} = 0$ for all $i \in N_1$ and all $j \in N_2$. In this case, it must be satisfied that $0 \geq U_k(e^{ij^m}) - \hat{q}_{ij^mk}$ and $0 \geq U_k(e^{ij^c}) - \hat{q}_{ij^ck} - c_k$. I can check it since $0 = \bar{w}_k \geq a_{\{i, j^m, k\}} - (\bar{u}_i + \bar{v}_{j^m}) = U_k(e^{ij^m}) - c_{j^m} - (\hat{q}_{ij^mk} - c_{j^m}) = U_k(e^{ij^m}) - \hat{q}_{ij^mk}$. Note that $\bar{w}_k = 0$ by Lemma 3, and $\bar{w}_k \geq a_{\{i, j^m, k\}} - (\bar{u}_i + \bar{v}_{j^m})$ by Lemma 2. Similarly, $0 = \bar{w}_k \geq a_{\{i, j^c, k\}} - (\bar{u}_i + \bar{v}_{j^c}) = U_k(e^{ij^c}) - c_i - c_k - (\hat{q}_{ij^ck} - c_i) = U_k(e^{ij^c}) - \hat{q}_{ij^ck} - c_k$.

Case 2: Consider that there exists a pair $(i, j^m) \in N_1 \times J^m$ such that $x^*_{ij^mk} = 1$ and $x^*_{i'j'k} = 0$ for all $(i', j') \in N_1 \times N_2$ with $(i', j') \neq (i, j^m)$. In this case, it must be satisfied that $U_k(e^{ij^m}) - \hat{q}_{ij^mk} \geq U_k(e^{i''j''}) - \hat{q}_{i''j''k}$ for all $(i'', j'') \in N_1 \times J^m$ with $(i'', j'') \neq (i, j^m)$, $U_k(e^{ij^m}) - \hat{q}_{ij^mk} \geq U_k(e^{i'''j'''}) - \hat{q}_{i'''j'''k} - c_k$ for all $(i''', j''') \in N_1 \times J^c$ and $U_k(e^{ij^m}) - \hat{q}_{ij^mk} \geq 0$. By Lemma 2, it can be checked since $U_k(e^{ij^m}) - \hat{q}_{ij^mk} = a_{\{i, j^m, k\}} + c_{j^m} - \hat{q}_{ij^mk} = (\bar{u}_i + \bar{v}_{j^m} + \bar{w}_k) - (\bar{u}_i + \bar{v}_{j^m}) = \bar{w}_k \geq a_{\{i'', j'', k\}} - (\bar{u}_{i''} + \bar{v}_{j''}) = U_k(e^{i''j''}) - c_{j''} - (\hat{q}_{i''j''k} - c_{j''}) = U_k(e^{i''j''}) - \hat{q}_{i''j''k}$, $U_k(e^{ij^m}) - \hat{q}_{ij^mk} = \bar{w}_k \geq a_{\{i''', j''', k\}} - (\bar{u}_{i'''} + \bar{v}_{j'''}) = U_k(e^{i'''j'''}) - c_{i'''} - c_k - (\hat{q}_{i'''j'''k} - c_{i'''}) = U_k(e^{i'''j'''}) - \hat{q}_{i'''j'''k} - c_k$ and $U_k(e^{ij^m}) - \hat{q}_{ij^mk} = \bar{w}_k \geq 0$.

Therefore, $(\hat{p}, \hat{q}, \hat{y})$ is a competitive equilibrium of the market with transaction via modern middlemen, and the competitive outcome at $(\hat{p}, \hat{q}, \hat{y})$ is $(\bar{u}, \bar{v}, \bar{w})$. ■

Next, I will state the main result. Theorem 1 shows that the law of one price holds in the assignment market with transaction via modern (resp. classical) middlemen when (i): the sum of the transaction costs of each seller and each buyer is relatively higher (resp. lower) than the matching cost of each modern middleman, and (ii): the number of modern (resp. classical) middlemen is not less than the number of potential assignments of goods.

Theorem 1 (*i*): *Let* $\alpha^{j^mj^c}_{ik} + c_i + c_k \geq c_{j^m}$ *for all* $(i, k) \in N_1 \times N_3$ *and all* $(j^m, j^c) \in J^m \times J^c$, *and* $n_m \geq \min\{n_1, n_3\}$. *Then, the core of* (N, V) *coincides with the set of competitive outcomes of the three-sided assignment market with transaction via modern middlemen.* (*ii*): *Let* $\alpha^{j^mj^c}_{ik} + c_i + c_k \leq c_{j^m}$ *for all* $(i, k) \in N_1 \times N_3$ *and all* $(j^m, j^c) \in J^m \times J^c$, *and* $n_c \geq \min\{n_1, n_3\}$. *Then, the core of* (N, V) *coincides with the set of competitive outcomes of the three-sided assignment market with transaction via* *classical middlemen.*

Proof. Immediate from Lemma 1 and Lemma 4. ■

The following example shows that a competitive equilibrium with transaction via (classical) middlemen does *not* necessarily exist even if the assumption of Theorem 1 is satisfied, that is, $\alpha_{ik}^{j^m j^c} + c_i + c_k \leq c_{j^m}$ for all $(i, k) \in N_1 \times N_3$ and all $(j^m, j^c) \in J^m \times J^c$, and $n_c \geq \min\{n_1, n_3\}$.

Example 1 *Let* $N_1 = \{i_1, i_2\}$, $N_3 = \{k_1, k_2\}$ *and* $N_2 = J^m \cup J^c$, *where* $J^m = \{j_1^m\}$ *and* $J^c = \{j_1^c, j_2^c\}$. *Let the utility functions and the costs of all agents be given by*

(*i*): $U_{i_1}(\omega_{i_1}) = U_{i_2}(\omega_{i_2}) = 0$;
(*ii*): $U_{k_1}(e^{i_2 j_2^c}) = U_{k_2}(e^{i_1 j_2^c}) = U_{k_2}(e^{i_2 j_1^c}) = U_{k_2}(e^{i_2 j_2^c}) = 1$;
(*iii*): $U_k(e^{ij}) = 0$ *otherwise*; *and*
(*iv*): $c_{i_1} = c_{i_2} = c_{k_1} = c_{k_2} = c_{j_1^m} = c_{j_1^c} = c_{j_2^c} = 0$.

The core of (N, V) *associated with this market is empty (See Quint (1991a)). By Theorem 1 (ii), there exists no competitive equilibrium.*

By Quint's theorem and Theorem 1 in the present study, the following result will be obtained.

Corollary 1 (*i*): *Let* $\alpha_{ik}^{j^m j^c} + c_i + c_k \geq c_{j^m}$ *for all* $(i, k) \in N_1 \times N_3$ *and all* $(j^m, j^c) \in J^m \times J^c$, *and* $n_m \geq \min\{n_1, n_3\}$. *Then, there exists a competitive equilibrium of the three-sided assignment market with transaction via modern middlemen if and only if matching for transaction in the corresponding market is socially optimal.* (*ii*): *Let* $\alpha_{ik}^{j^m j^c} + c_i + c_k \leq c_{j^m}$ *for all* $(i, k) \in N_1 \times N_3$ *and all* $(j^m, j^c) \in J^m \times J^c$, *and* $n_c \geq \min\{n_1, n_3\}$. *Then, there exists a competitive equilibrium of the three-sided assignment market with transaction via classical middlemen if and only if matching for transaction in the corresponding market is socially optimal.*

This corollary implies that there does *not* necessarily exist a competitive equilibrium with transaction via modern (resp. classical) middlemen even if the transaction costs of each seller and each buyer are sufficiently higher (resp. lower) than the matching cost of each modern middleman.

Next, the following example shows that a competitive outcome of each (modern) middleman is *not* necessarily zero if the assumption of Corollary 1 is satisfied, that is, $\alpha_{ik}^{j^m j^c} + c_i + c_k \geq c_{j^m}$ for all $(i, k) \in N_1 \times N_3$ and all $(j^m, j^c) \in J^m \times J^c$, and $n_m \geq \min\{n_1, n_3\}$.

Example 2 *Let* $N_1 = \{i_1, i_2\}$, $N_3 = \{k_1, k_2\}$ *and* $N_2 = J^m \cup J^c$, *where* $J^m = \{j_1^m, j_2^m\}$ *and* $J^c = \{j_1^c, j_2^c\}$. *Let the utility functions of all agents be given by*

(*i*): $U_{i_1}(\omega_{i_1}) = U_{i_2}(\omega_{i_2}) = 0$; (*ii*): $U_{k_1}(e^{i_1 j_1^m}) = 6$;
(*iii*): $U_{k_2}(e^{i_2 j_2^m}) = 8$; *and* (*iv*): $U_k(e^{ij}) = 1$ *otherwise.*

Let the costs of all agents be given by

(*i*): $c_{i_1} = c_{i_2} = 1/2$; (*ii*): $c_{j_1^m} = c_{j_2^m} = 1$;
(*iii*): $c_{j_1^c} = c_{j_2^c} = 0$; *and* (*iv*): $c_{k_1} = c_{k_2} = 1/2$.

There exists only one integral solution for (P) *associated with the above market, then there exists* $\hat{y} \in Y \subseteq \mathbb{Z}_+^{18}$ *such that* $\hat{y}_{i_1 j_1^m k_1} = \hat{y}_{i_2 j_2^m k_2} = 1$ *and all other components of* $\hat{y}$ *are* 0. *I have that*

(*i*): $V(N) = 12$;
(*ii*): $V(S) = 5$ *if* $\{i_1, j_1^m, k_1\} \in \rho s$ *and if* $\{i_2, j_2^m, k_2\} \notin \rho s$;
(*iii*): $V(S) = 7$ *if* $\{i_2, j_2^m, k_2\} \in \rho s$ *and if* $\{i_1, j_1^m, k_1\} \notin \rho s$; *and*

(iv): $V(S) = 0$ *otherwise.*

The core of (N, V) *is given by the set of* $(\bar{u}, \bar{v}, \bar{w}) = ((\bar{u}_i)_{i \in N_1}, (\bar{v}_j)_{j \in N_2}, (\bar{w}_k)_{k \in N_3})$ *such that* (i): $\bar{u}_i \geq 0$ *for all* $i \in N_1$, $\bar{v}_j \geq 0$ *for all* $j \in N_2$, *and* $\bar{w}_k \geq 0$ *for all* $k \in N_3$; (ii): $\bar{u}_{i_1} + \bar{v}_{j_1^m} + \bar{w}_{k_1} = 5$; *and* (iii): $\bar{u}_{i_2} + \bar{v}_{j_2^m} + \bar{w}_{k_2} = 7$.

Let $(\hat{u}, \hat{v}, \hat{w})$ *be an element of the core of* (N, V) *such that* $\hat{u}_{i_1} = 2$, $\hat{u}_{i_2} = 3$, $\hat{v}_{j_1^m} = \hat{v}_{j_2^m} = 1$, $\hat{v}_{j_1^c} = \hat{v}_{j_2^c} = 0$, $\hat{w}_{k_1} = 2$ *and* $\hat{w}_{k_2} = 3$. *By Theorem 1*(i), $(\hat{u}, \hat{v}, \hat{w})$ *is a competitive outcome of the market* (N_1, N_2, N_3) *at competitive equilibrium price* $(\hat{p}, \hat{q})$ *such that*

$$(i): \hat{p}_{i_1 j_1^m} = \hat{p}_{i_1 j_2^m} = 2,\ \hat{p}_{i_2 j_1^m} = \hat{p}_{i_2 j_2^m} = 3,\ \hat{p}_{i_1 j_1^c} = \hat{p}_{i_1 j_2^c} = 5/2;\ \hat{p}_{i_2 j_1^c} = \hat{p}_{i_2 j_2^c} = 7/2;$$

and

$$(ii): \hat{q}_{i_1 j_1^m k_1} = \hat{q}_{i_1 j_2^m k_1} = \hat{q}_{i_1 j_1^m k_2} = \hat{q}_{i_1 j_2^m k_2} = 4;$$
$$\hat{q}_{i_2 j_1^m k_1} = \hat{q}_{i_2 j_2^m k_1} = \hat{q}_{i_2 j_1^m k_2} = \hat{q}_{i_2 j_2^m k_2} = 5;$$
$$\hat{q}_{i_1 j_1^c k_1} = \hat{q}_{i_1 j_2^c k_1} = \hat{q}_{i_1 j_1^c k_2} = \hat{q}_{i_1 j_2^c k_2} = 5/2;$$
$$\hat{q}_{i_2 j_1^c k_1} = \hat{q}_{i_2 j_2^c k_1} = \hat{q}_{i_2 j_1^c k_2} = \hat{q}_{i_2 j_2^c k_2} = 7/2.$$

4 Conclusion

In this paper, I have proved that the core coincides with the set of competitive equilibrium allocations in an assignment market with middlemen. Using this core equivalence theorem, I have characterized a sufficient condition for the existence of a competitive equilibrium in this market. Intuitively speaking, this condition is based on the two factors: (a): free entry of middlemen; (b): socially optimal matching for transaction. From my observation in the present study (see Example 1), it follows that there may be no competitive equilibrium if a three-sided assignment market has only factor (a) without factor (b). This theoretical finding leads to the following economic implication: In order to design healthy markets with middlemen, free-entry market policy is insufficient, and market policy toward socially optimal matching is essential for a free-entry situation of middlemen.

Since it is known that the existence of a solution of an integer program is directly relevant to discrete convex analysis (e.g., Murota, 2003), the existence problem of a competitive equilibrium in the present market model may be resolved by some techniques in discrete convex analysis. Therefore, whether or not the sufficient condition can be characterized from a viewpoint of discrete convex analysis may deserve investigation, which I leave to the future research.

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Notes

1. There can be found the literature pointing the importance of transaction costs in competitive market models, e.g., Ostroy and Starr (1990).
2. This assumption is different from the underlying assumption of search theoretic models.
3. In Oishi and Sakaue (2009), only a modern type of middlemen is incorporated into the assignment market model in Shapley and Shubik (1972). In the present study, on the other hand, both modern middlemen and classical middlemen, and their cost structure are incorporated into the Shapley and Shubik's model.
4. In the case of the Maghribi traders, for example, c_i means transportation costs which seller i incurs to perform transaction of the good ω_i between a buyer and herself.
5. The matching cost of a real estate broker in housing markets may be regarded as the cost of his repairing a house based on a buyer's request.
6. In the case of the Maghribi traders, c_k

means measuring costs which buyer k incurs to identify quality of the good ω_i for herself.

References

[1] Blume, L.E., D. Easley, J. Kleinberg, and E. Tardos, 2009. Trading networks with price-setting agents. *Games and Economic Behavior*, Volume 67, Issue 1, 36-50.

[2] Crawford, V.P. and E.M. Knoer, 1981. Job matching with heterogeneous firms and workers. *Econometrica* , Volume 49, Issue 2, 437-450.

[3] Dantzig, G.B., 1963. *Linear Programming and Extensions*. Princeton University Press, Princeton, NJ.

[4] Gehring T., 1993. Intermediation in search markets. *Journal of Economics and Management Strategy*, Volume 2, Issue 1, 97-120.

[5] Greif, A., 1989. Reputation and coalitions in medieval trade: Evidence on the Maghribi traders. *Journal of Economic History*, Volume 49, Issue 4, 857-882.

[6] Johri, A. and J. Leach, 2002. Middlemen and the allocation of heterogeneous goods. *International Economic Review*, Volume 43, Issue 2, 347-361.

[7] Kaneko, M., 1976. On the core and competitive equilibria of a market with indivisible goods. *Naval Research Logistics Quarterly*, Volume 23, Issue 2, 321-337.

[8] Kaneko, M., 1982. The central assignment game and the assignment markets. *Journal of Mathematical Economics*, Volume 10, Issues 2-3, 205-232.

[9] Kaneko, M. and M.H. Wooders, 1982. Cores of partitioning games. *Mathematical Social Sciences*, Volume 3, Issue 4, 313-327.

[10] Kelso, A.S. and V.P. Crawford, 1982. Job matching coalition formation and gross substitutes. *Econometrica*, Volume 50, Issue 6, 1483-1504.

[11] McMillan, J., 2002. *Reinventing the Bazaar: A Natural History of Markets*. W. W. Norton & Company, New York.

[12] Murota, K., 2003. *Discrete Convex Analysis*. Society for Industrial and Applied Mathematics, Philadelphia.

[13] Oishi, T. and S. Sakaue, 2009. Competitive equilibria in a market for indivisible commodities with middlemen. *Keio University/Kyoto University Global COE Discussion Paper Series*, DP2008-016.

[14] Ostroy, J.M. and R.S. Starr, 1990. The transaction role of money. in B. M. Friedman and F. H. Hahn (ed.), *Handbook of Monetary Economics*, Volume I, Chapter 1, 3-62, Elsevier Science Publishers, Amsterdam.

[15] Quint, T., 1991a. The core of an m-sided assignment game. *Games and Economic Behavior*, Volume 3, Issue 4, 487-503.

[16] Quint, T., 1991b. Necessary and sufficient conditions for balancedness in partitioning games. *Mathematical Social Sciences*, Volume 22, Issue 1, 87-91.

[17] Quinzii, M., 1984. Core and competitive equilibria with indivisibilities. *International Journal of Game Theory*, Volume 13, Issue 1, 41-60.

[18] Rubinstein, A. and A. Wolinsky, 1987. Middlemen. *Quarterly Journal of Economics*, Volume 102, Issue 3, 581-593.

[19] Shapley, L.S. and M. Shubik, 1972. The assignment game I: The core. *International Journal of Game Theory*, Volume 1, Issue 1, 111-130.

[20] Yang, Z., 2003. A competitive market model for indivisible commodities. *Economics Letters*, Volume 78, Issue 1, 41-47.

[21] Yano, M., 2008. Competitive fairness and the concept of a fair price under Delaware Law on M&A. *International Journal of Economic Theory,* Volume 4, Issue 2, 175- 190.

[22] Yavaş, A., 1992. Marketmakers versus matchmakers. *Journal of Financial Intermediation*, Volume 2, Issue 1, 33-58.

[23] Yavaş, A., 1994. Middlemen in bilateral search markets. *Journal of Labor Economics*, Volume 12, Issue 3, 406-429.